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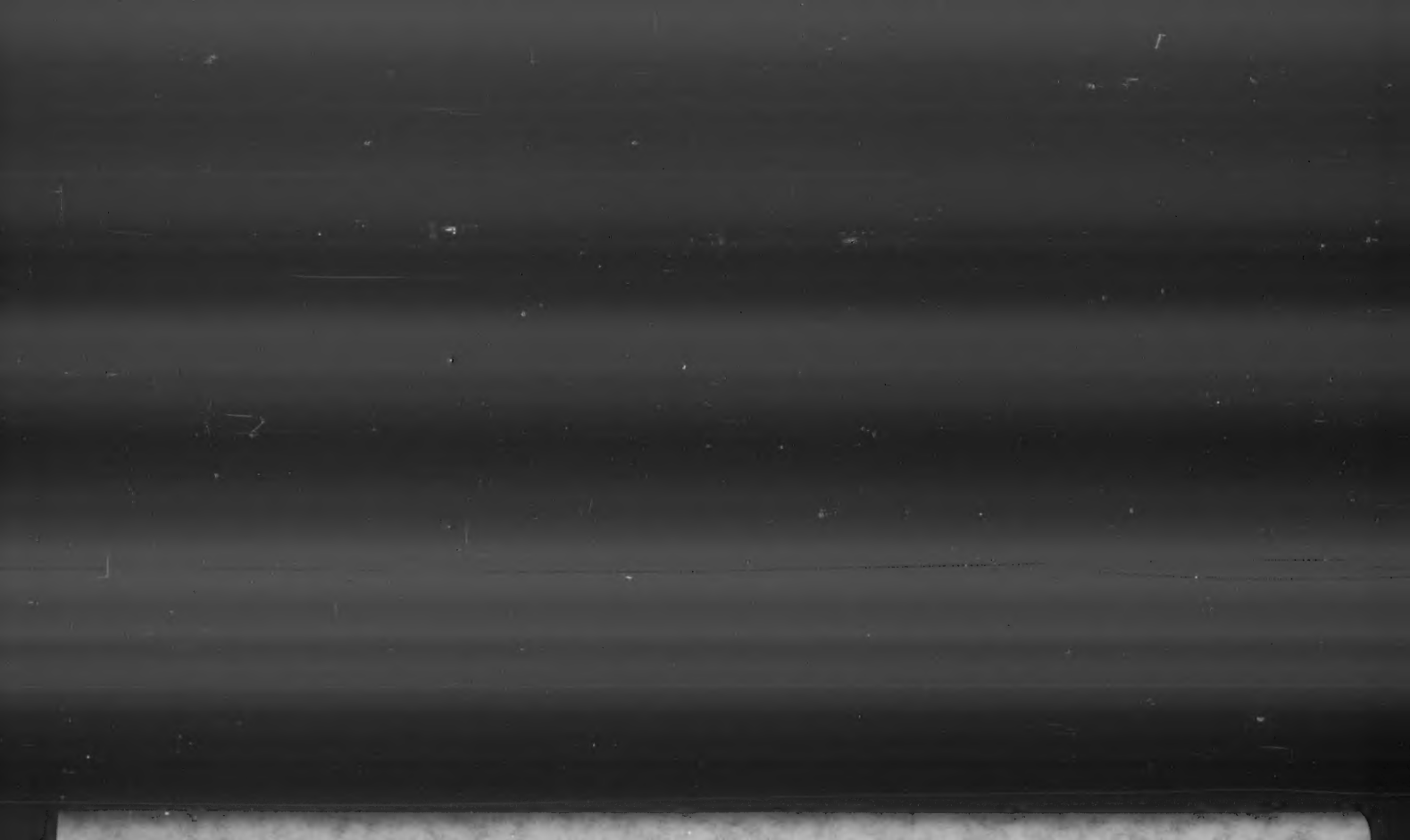

SOME REMARKS ON SET THEORY

P. ERDÖS

The present note contains a few disconnected remarks on the theory of sets.

It is well known that the addition of ordinal numbers does not satisfy the the law of commutativity, for example, $1+\omega \neq \omega+1$. Let now n be a finite number. Denote by $f(n)$ the maximum number of different ordinals we can obtain by adding in all possible ways n ordinals. We prove the following theorem.

THEOREM I. *We have*

$$f(n) = \max_{k \leq n-1} (k2^{k-1} + 1) f(n-k). \tag{1}$$

In fact, $f(2)=2$, $f(3)=5$, $f(4)=13$, $f(5)=33$, $f(6)=81$, $f(7)=193$, $f(8)=449$, $f(9)=1089$, $f(10)=2673$, $f(11)=6561$, $f(12)=15633$, $f(13)=37249$, $f(14)=88209$, $f(15)=216153$, $\cdots$, *and for* $x \geqq 3$, $f(5x+1)=81^x$, $f(5x+2)=193 \cdot 81^{x-1}$, $f(5x+3)=193^2 81^{x-2}$, $f(5x+4)=193^3 81^{x-3}$, $f(5x+5)=33 \cdot 81^x$. *Thus for* $n \geqq 21$

$$f(n) = 81 f(n-5). \tag{2}$$

Let there be given n ordinals $\alpha_1, \alpha_2, \cdots, \alpha_n$. It is well known that every ordinal can be written uniquely as the sum of indecomposable ordinals. (An ordinal is said to be indecomposable if it is not the sum of two smaller ordinals.) Denote by $\phi(\alpha)$ the largest of these indecomposable ordinals belonging to α. ($\phi(\alpha)$ may have a coefficient c in the decomposition of α.) Put $\gamma = \min_{i \leq n} \phi(\alpha_i)$, and assume that there are k α's with $\phi(\alpha_i)=\gamma$. Denote these α's by $\alpha_1, \alpha_2, \cdots, \alpha_k$. If in the sum $\alpha_{i_1}+\alpha_{i_2}+\cdots+\alpha_{i_n}$, $i_1, i_2, \cdots, i_n$ a permutation of $1, 2, \cdots, n$, none of the α_i, $i \leq k$ appear at the end, they get absorbed in the following summands, and we get exactly $f(n-k)$ different sums. Assume next that exactly r of the α_i's, $r \leq k$, appear at the end of $\alpha_{i_1}+\alpha_{i_2}+\cdots+\alpha_{i_n}$. Put

$$\alpha_i = 2^i\gamma + \delta_i, \quad \delta_i < \gamma, \quad \delta_i \neq \delta_j, \qquad \text{for } i, j \leq k. \tag{3}$$

We then have

$$\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_n} = \beta + (2^{i_n} + 2^{i_{n-1}} + \cdots + 2^{i_{n-r+1}})\gamma + \delta_{i_n} \tag{4}$$

where $\beta \geqq \omega\gamma$. (All the δ's except δ_{i_n} get absorbed.) Now $2^{i_n}+2^{i_{n-1}}+\cdots+2^{i_{n-r+1}}$ can be chosen in $C_{k,r}$ ways and δ_{i_n} in r ways. Thus

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the number of sums is $rC_{k,r}f(n-k)$. (The $k-r$ α's with $\phi(\alpha)=\gamma$ not appearing at the end get absorbed.) Summing for r, we obtain

$$(5) \qquad f(n) \geqq \left[1 + \sum_{r=1}^{k} rC_{k,r}\right] f(n-k) = (k2^{k-1}+1)f(n-k).$$

Hence clearly

$$(6) \qquad f(n) = \max_{k \leqq n-1} (k2^{k-1}+1)f(n-k)$$

(since it clearly follows from our proof that our choice of the α's gives the maximum number of different summands). This proves (1).

We obtain from (1) by a simple computation that for $n \leqq 20$ the value of $f(n)$ is given by Theorem I. The rest of Theorem I is easily proved by induction, we have to use that $(k2^{k-1}+1)^{1/k}$ (k integer) increases for $k \leqq 5$ and decreases for $k \geqq 5$. We suppress the details since they can easily be given and depend only on numerical estimates. Thus the proof of Theorem I is complete.

Mr. Spanier remarked that the number of different products one can obtain from n ordinals is $n!$. It suffices to choose $\alpha_1=\omega+1$, $\alpha_2=\omega+2, \cdots, \alpha_n=\omega+n$. A simple computation shows that

$$\alpha_{i_1} \cdot \alpha_{i_2} \cdots \alpha_{i_n} = \omega^n + i_n\omega^{n-1} + \cdots + i_1$$

where $i_1, i_2, \cdots, i_n$ is any permutation of $1, 2, \cdots, n$.

Let X be a set of power m. Letters $a, b, \cdots$ denote subsets of X, $A, B, \cdots$, sets of subsets of X. A and B are defined (by Lusin) to be orthogonal if for any $a \in A$, $b \in B$, $a \cap b$ has power less than m. The orthogonal sets are said to be separable if there exist c and d with $c \cap d$ empty and such that for every $a \in A$, $b \in B$, $a \subset c \cup a'$, $b \subset d \cup b'$, where the power of a' and b' is less than m.

Lusin[1] proves with the aid of the axiom of choice that if $m=\aleph_0$ there exists two orthogonal sets which are not separable. We shall give a very simple proof of this result for all m, which for $m=\aleph_0$ will be independent of the axiom of choice.

Let A consist of m disjoint sets of power m and B consist of all the sets which intersect the sets of A in not more than one point. Clearly A and B are orthogonal, but they clearly are not separable, for every set c which intersects all the sets of A contains a set of B. For $m=\aleph_0$ this proof is independent of the axiom of choice, but in the general case the equality $m^2=m$ is used and this is equivalent to the axiom of choice.[2]

[1] C. R. (Doklady) Acad. Sci. URSS. vol. 40 (1943) pp. 175–178.

[2] Tarski, Fund. Math. vol. 7, pp. 147–154.

Lusin[1] also proves that if both A and B contain only countably many sets, then they are separable. His proof generalizes to the case when m is a regular number and both A and B contain only m sets. We give the proof only for the sake of completeness.

Suppose A consists of the sets $\{a_\alpha\}$ and B of the sets $\{b_\alpha\}$. Put

$$c = \bigcup_\alpha \left(a_\alpha - \sum_{\beta \leq \alpha} b_\beta\right), \qquad d = \bigcup_\alpha \left(b_\alpha - \sum_{\beta \leq \alpha} a_\beta\right).$$

Clearly c and d separate A and B.

The proof breaks down if m is singular. We shall give an example which shows that the theorem is not always true for singular m. Put

$$m = \aleph_0 + 2^{\aleph_0} + 2^{2^{\aleph_0}} + \cdots.$$

Let A consist of countably many disjoint sets a_i of power m. Write $a_i = n_0^{(i)} + n_1^{(i)} + \cdots$ where card $(n_0^{(i)}) = \aleph_0$, card $(n_1^{(i)}) = \aleph_1$, $\cdots$ (card a denotes the cardinal number of a) and $n_k^{(i)} \cap n_i^{(i)} = 0$. B consists of all the sets b of the form $\bigcup_i n_{f(i)}^{(i)}$ where $f(i)$ is any function of i (clearly card $B = 2^{\aleph_0} < m$). A and B are clearly orthogonal, but a simple argument shows that they are not separable.

If n is the smallest cardinal number cofinal to m and A and B contain not more than n sets, then it is easy to see that if A and B are orthogonal, they are also separable.

The orthogonal sets A and B are said to be complete (Lusin) if we can add no set either to A or to B without destroying orthogonality, that is, if X is the set of integers and A consists of all the subsets containing only a finite number of even numbers and B of all sets containing only finitely many odd numbers.

A and B are called k-orthogonal if for any $a \in A$ and $b \in B$, card $(a \cap b) < \aleph_k$. k-completeness can be defined in the obvious way. We shall prove the following theorem.

THEOREM II. *The cardinal number N of the k-complete orthogonal pairs equals*

$$N = 2^{m^{\aleph_k}}.$$

We shall assume the generalized continuum hypothesis $2^{\aleph_k} = \aleph_{k+1}$. Tarski[2] proved (by using the generalized hypothesis of the continuum) that there exist $m^{\aleph_k}$ subsets of X(card $X = m$) such that the intersection of any two has power less than $\aleph_k$. Denote such a set of subsets by C. Split C in an arbitrary way into the union of two sets of subsets A' and B'. This can clearly be done in $2^{m^{\aleph_k}}$ ways. Clearly

[2] Ibid. vol. 12, pp. 186–206 and vol. 14, pp. 205–216.

A' and B' are k-orthogonal. It immediately follows from the axiom of choice that they can be extended to the k-complete orthogonal pair A and B, and a simple argument shows that to different A' and B' correspond different A and B. This shows that

$$N \geqq 2^{m^{\aleph_k}}.$$

Let now A' be any set of subsets of X. There clearly exists a maximal set of subsets of X, B say, which is k-orthogonal to A'. We shall prove that there exists a subset A'' of A' of power $\leqq m^{\aleph_k}$ which determines the same set B. Consider all the subsets of power $\aleph_k$ of the sets $a \in A'$. The power of these sets is clearly not greater than $m^{\aleph_k}$. To each of these sets select an arbitrary $a_\alpha \in A'$ which contains it, and let the sets of A'' be all these a_α. Clearly if B is k-orthogonal to A'' then it is also k-orthogonal to A' (and vice versa), which completes the proof of our statement. But then a simple argument shows that the number of complete k-orthogonal pairs is not greater than

$$(2^m)^{m^{\aleph_k}} = 2^{m^{\aleph_k}}$$

which completes the proof of Theorem II. In the second part of our proof we clearly did not use the continuum hypothesis.

It follows from Theorem II that the number of complete orthogonal pairs is 2^{2^m}. The number of complete orthogonal and separable pairs is clearly only 2^m, which again shows that there are orthogonal pairs which are not separable.

Without using the continuum hypothesis it seems to be very hard to prove that the number of complete orthogonal pairs is greater than 2^m. But if $m = \aleph_0$ this is quite easy, since it is well known that there exist $2^{\aleph_0}$ sets of integers such that the intersection of any two is finite. It suffices to let a_α consist of the integers $2^n + [n^\alpha]$ where α is an arbitrary positive real number.

THEOREM III. *Let S be any infinite subset of k-dimensional Euclidean space.* Card $S = m$. *Then there exists a subset S_1 of S,* card $S_1 = m$, *such that all the distances between any two points of S_1 are different.*

REMARK. We are not going to assume the continuum hypothesis. In fact the proof will be complicated only because we cannot exclude the possibility that m is singular.

We use induction with respect to the dimension k. We slightly strengthen the statement of our theorem. In fact we prove: Let the set S (card $S = m$) be situated on n ($n < m$) k-dimensional hyperplanes or hyperspheres. Then there exists a subset S_1 of S, card $S_1 = m$, such that all the distances between any two points of S_1 are different.

First we prove this for $k=1$. Put $S=L+C$, where L is the subset of S situated on lines and C the subset situated on circles. Assume first card $L=m$. Denote by M_L the set of lines containing L. By assumption card $M_L \leqq n<m$. Let L_1 be a subset of L, such that any line bisecting the distance between any two points of L_1 is not in M_L, and L_1 is maximal with respect to this property (it clearly follows from the axiom of choice that such an L_1 exists, possibly it is empty). We prove card $L_1=m$. For if not assume that card $L_1=r<m$, then by definition of L_1, to every point x of $L-L_1$ there exists a point y of L_1 such that the perpendicular bisector of x and y is a line of M_L. But a point y of L_1 and a line l of M_L uniquely determines a point x of $L-L_1$ such that the perpendicular bisector of the segment $[x, y]$ is l. But since card $L_1<m$ and card $M_L<m$ this would imply that $L-L_1 \leqq m$, and hence card $L<m$, which is not true. Thus we prove that card $L_1=m$. Let now L_2 be a maximal subset of L_1 with the property that all distances between points of L_2 are different. Let card $L_2=t$. If $t<m$, then all the points of L_1-L_2 must lie on t circles or lines. But this is impossible since the lines cannot coincide with any of the lines of M_L, therefore any of these lines can intersect any line of M_L in at most one point. Therefore each of these lines can contain at most card M_L points of L_1-L_2, and each of the circles can contain also at most card M_L points of L_1-L_2. But this would mean that card $(L_1-L_2) \leqq t \cdot$ card $M_L<m$ which is not the case. Thus $t=m$, which completes the case card $L=m$. Assume next card $C=m$. Denote by M_c the set of circles containing C, put card $M_C=r<m$, denote further by O the set of centers of the circles of M_C, and consider the set $C-O$. Denote by C_1 a maximal subset of $C-O$ with the property that all distances between points of C_1 are different. Then it is easy to see that card $C_1=m$, which completes the proof in case $k=1$.

Assume now that our theorem is true for $k-1$, and we shall prove it for k. Suppose then that S lies on $n<m$ k-dimensional hyperplanes and hyperspheres. By the same argument as used in the previous pages we can find n $(k-1)$-dimensional hyperplanes and hyperspheres which contain m points of S, and by the induction hypothesis this completes the proof of Theorem III. In case m is a regular number it is easy to give a very much simpler proof.

Assume that m is regular and that S is a set of power m in k-dimensional Euclidean space. Then we can select a subset S_1 of S of power m such that for $r=1, 2, \cdots, k$ the volume of any two r-dimensional nondegenerate simplices is different. The proof is similar to the case for $r=1$. If m is singular this result is false. Take $m=\aleph_\omega$, and denote

by $l_1, l_2, \cdots$ a countable set of parallel lines. Let l_k contain $\aleph_k$ points of S. Clearly any subset S_1 of S of power $\aleph_1$ contains two nondegenerate triangles of the same area.

If the set S is in Hilbert space our theorems of the previous pages do not generalize. In fact Oxtoby and I constructed a set of power $\aleph_1$, such that the distance between any two points is rational. We do not give the construction here. At present we cannot decide whether there exists in Hilbert space such a set of power c. (Added in proof: Kakutani and I found a simple example of such a set.)

One can ask the following question: Is it possible to split the n-dimensional Euclidean space into countably many disjoint sets, such that in each set any two distances between any two points should be different. For $n=1$ this is known,[4] and I do not know the answer for $n>1$.

The problems we just considered have some interest also for finite sets. Let there be given n points (n finite) on a line. Then we can select $c\cdot n^{1/3}$ points among them, such that all the distances are different. It is probable that $c\cdot n^{1/3}$ can be replaced by $c\cdot n^{1/2}$. If the n points are in k-dimensional space, we can select $c\cdot n^{1/f(k)}$ points among them such that all the distances are different. I do not know the exact value of $f(k)$.

Dénes König's[5] book on graphs contains the following theorem: Let G be a graph of order m (m is an infinite cardinal) and any two vertices are connected by less than n edges, and we assume $n<m$. Then G is the product of linear factors. Let us first explain the terminology: The order of a point is the cardinal number of edges of its star (the star of a vertex is the collection of edges incident with it). A graph is said to be of order m, if every vertex of it has order m. Let Ω_k be the initial number belonging to the cardinal number m. G is the product of linear factors if we can make correspond to each edge of G an ordinal number $\beta<\Omega_k$, such that the star of every vertex contains one and only one edge to which the ordinal β, $\beta=1, 2, \cdots$, corresponds. König[5] raises the question whether the condition "connected by less than n edges, and $n<m$" can be replaced by "connected by less than m edges." The proof given in his book shows that the answer is affirmative in case m is a regular number. We shall show that the answer is negative for singular numbers. Let the vertices of G be the points $\{a_i\}$, $i=1, 2, \cdots$, and $\{b_\alpha\}$, $\alpha<\Omega_1$. The vertices a_i and a_j, $i<j$, are connected by $\aleph_j$ edges, and b_α and a_i are connected by $\aleph_i$ edges. Clearly G is of order $\aleph_\omega$ and any two vertices are con-

[4] P. Erdös and S. Kakutani, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 457–460.

[5] *Theorie der endlichen und unendlichen Graphen*, pp. 220–223.

nected by less than $\aleph_\omega$ edges. Nevertheless G is not the product of linear factors, in fact G has no linear factor at all. (A linear factor is a subgraph of order 1 containing all the vertices of G.) Clearly any linear factor would have all the b's as vertices, but then at least one a would have to have order $\aleph_1$, which is impossible; this contradiction proves our assertion. It would be easy to construct an analogous counterexample for every other singular number.

Now we prove the following theorem.

THEOREM IV. *Let G be a graph of order m, where every vertex is connected (by an edge) to at least m different vertices. Then G is the product of linear factors.*

REMARK. This theorem is clearly a generalization of the theorem given in König's book.[5] My original proof was very complicated. Hajós found the following very much simpler proof:

Let Ω_k be the initial number belonging to m, $\{a_\alpha\}$, $\alpha<\Omega_k$, be the vertices of G and $\{e_\beta\}$, $\beta<\Omega_k$, the edges of G. We construct the factors of G by transfinite induction. Let $0<\gamma<\Omega_k$, and suppose that for every $\delta<\gamma$ we have already found a linear factor. Then we construct the γth linear factor as follows: Let e_ρ be the edge of smallest index which has not been used in any of the previous linear factors (in other words every $e_{\rho'}$, with $\rho'<\rho$, occurs in some previous linear factor). Let then e_ρ be the first edge of our γth linear factor. We construct the γth factor by transfinite induction. Suppose we have already constructed a subgraph $G_\gamma(\alpha)$ of order 1 of G containing e_ρ and also all the vertices of index less than α of G. Clearly $G_\gamma(\alpha)$ has less than m vertices. By our assumption a_α is connected to at least m different vertices. Thus it is connected to m different vertices not in $G_\gamma(\alpha)$. In the factors of index less than γ we clearly used less than m (in fact card γ) edges all emanating from a_α. Thus there remains an unused edge of G which connects a_α to a vertex not in $G_\gamma(\alpha)$, this edge will be in our γth factor. This construction gives our γth factor for every $\gamma<\Omega_k$ and clearly by construction every edge of G occurs in one of the factors once and only once, q.e.d. This proves Theorem IV.

The following problem is due to Turán[6] (oral communication):

Let card $S=c$, to every $a\in S$ there corresponds a finite subset $f(a)$ of S. We assume that a is not contained in $f(a)$. Two elements a and b

[6] The problem was originally raised by Turán in connection with a problem on interpolation and solved by G. Grünwald in case card $S=\aleph_0$. For the literature of the problem see, for example, P. Erdös, *Some set-theoretical properties of graphs*, Universidad Nacional de Tucuman Revista, 1942, also footnote 7.

are called independent if $a \notin f(b)$ and $b \notin f(a)$. A subset S' of S is said to be independent if any two elements of it are independent or if $S' \cap f(S')$ is empty. Turán's question was: does there always exist an infinite independent set? G. Grünwald showed that the answer is affirmative. Later Lázár showed that there exists an independent set of power c. Sierpinski and Ruzievicz raised the following general question: Let S be a set of power m, and let $n < m$, an arbitrary cardinal number. To every $a \in S$ there corresponds a subset $f(a)$ of S satisfying $a \notin f(a)$ and $f(a) < n$. Does there always exist an independent set S' with card $S' = m$? This has been proved if m is a regular number, or if m is the countable sum of smaller cardinals.[7] We shall prove, assuming the generalized continuum hypothesis, that the answer is always affirmative (Theorem V).

If we replace the condition card $(f(a)) < n < m$ by card $(f(a)) < m$, then in general we do not even have two independent points. Let Ω_k be the initial number belonging to m, and α_ξ the set of ordinals less than Ω_k. Define $f(\alpha_\xi)$ as the collection of all $\beta < \alpha_\xi$. Clearly no two elements are independent.

First I give a very simple proof of Theorem V in case m is a regular number. This proof is due to D. Lázár, and has been communicated to me orally. Assume that the theorem is false, that is, the power of every independent set is less than m. Let S_1 be a complete set of independent elements, that is, $S_1 \cap f(S_1)$ is empty and if $a \notin S_1$, then $(a \cup S_1) \cap f(a \cup S_1)$ is not empty. Consider $S - S_1 - f(S_1)$, and let S_2 be a maximal independent subset of it, and consider $S - S_1 - S_2 - f(S_1) - f(S_2)$. Continue this process for all ordinals $\alpha < \Omega_l$, where Ω_l is the initial number belonging to $n = \aleph_l$. Since m is a regular number, the power of $\bigcup_\alpha (S_\alpha \cup f(S_\alpha))$ is less than m. Thus the set $S - \bigcup_\alpha (S_\alpha \cup f(S_\alpha))$ is not empty. Let a be an element of this set. Because of the maximal property of the sets S_α, $f(a)$ must intersect each of the S_α's (by our construction $a \notin f(S_\alpha)$, thus the last statement is a consequence of the fact that a and S_α are not independent). But this would mean that card $(f(a)) \geqq n$, an evident contradiction. Clearly this proof also works if m is singular but is not the sum of n smaller cardinal numbers.

Now we prove our main theorem. Let card $S = m$, m singular. Put $S = \bigcup_\alpha S_\alpha$, $\alpha < \Omega_r$. Card $(S_\alpha) = m_\alpha$, $m_\alpha > m_\beta$ for $\alpha > \beta$. Let Ω_k be the initial number belonging to m and Ω_r the smallest ordinal number cofinal to Ω_k. We can clearly assume that each m_α is regular (every singular number is the sum of fewer, smaller regular numbers). Also

[7] Sophie Piccard, Fund. Math. vol. 29, pp. 5–9. See also Comptes Rendus des Séances de la Société des Sciences et des Léttres de Varsovie vol. 30 (1937).

we can assume that $m_1 > n^{\aleph_r}$ and $m_1 > \text{card } (\Omega_r)$. By the theorem proved in the previous pages, there exists for every S_α a set $S'_\alpha \subset S_\alpha$, such that $S'_\alpha \cap f(S'_\alpha)$ is empty and card $(S'_\alpha) = m_\alpha$. Omit for $\beta > \alpha$ all the elements of $f(S'_\alpha)$ from S'_β. Thus we get the sets S''_α of power m_α, such that

$$f(S''_\alpha) \cap \left(\bigcup_{\beta \geqq \alpha} S''_\beta \right) = 0 \qquad (0 \text{ stands for the empty set}). \tag{7}$$

We want to construct sets $S'''_\alpha \subset S''_\alpha$ of power m_α which further satisfy

$$f(S'''_\alpha) \cap \left(\bigcup_{\beta < \alpha} S'''_\beta \right) = 0. \tag{8}$$

But then clearly

$$f\left(\bigcup_\alpha S'''_\alpha \right) \cap \left(\bigcup_\alpha S'''_\alpha \right) = 0. \tag{9}$$

Thus the set $\bigcup_\alpha S'''_\alpha$ is independent and clearly of power m. Thus we only have to construct S'''_α. We shall use transfinite induction. Let as before Ω_l be the initial number belonging to n and let N be any set with card $N \geqq n^{\aleph_r}$. We construct a ramification system belonging to N as follows: Consider the disjoint sets

$$\begin{gathered} N^{(\Omega_r)}_{i_1, \cdots, i_j, \cdots} \; i_j < \Omega_l, \; j < \Omega_r; \\ N = \bigcup_{i_1, \cdots, i_j, \cdots} N^{(\Omega_r)}_{i_1 \cdots}; \text{ card } (N^{(\Omega_r)}_{i_1 \cdots}) = \text{card } N. \end{gathered} \tag{10}$$

Clearly there are $n^{\aleph_r} \leqq N$ such sets, thus since $N^2 = N$ this is clearly possible. The sets (10) will be the Ω_rth (and last) column of the ramification system. The sets of the kth, $1 \leqq k < \Omega_r$, column we define as follows: ($i_1, i_2, \cdots, i_j, \cdots$ again run through the ordinals less than Ω_l and $j < k$)

$$N^{(k)}_{i_1 \cdots i_j \cdots} = \bigcup_{i_k \cdots} N^{(\Omega_r)}_{i_1 \cdots i_j \cdots i_k \cdots}, \tag{11}$$

this means that $N^{(k)}_{i_1} \ldots$ is the union of all the sets of (10) whose indices agree with it for $j < k$. (The 0th column $N^{(1)}$ is N.) We will denote this ramification system by $R(N)$.

Consider now $R(S'_1)$. Let a be any element of S'_2. Since card $(f(a)) < n$ there must exist an $(S'_1)^{(2)}_i$ (that is, a set of the first column of $R(S'_1)$) such that $f(a) \cap S'_1)^{(2)}_i = 0$. Now since m_2 (card $S''_2 = m_2$) is regular (thus not the sum of n sets of power $< m_2$) there exists a set

$S_2'' \subset S_2'$ with card $S_2'' = m_2$ and an index $i_{(1)}^{(1)} < \Omega_l$ so that

$$f(S_{2'}'') \cap (S_1'')_{i_1(1)}^{(2)} = 0.$$

Consider now $R(S_2'')$. Let a be any element of S_3''. As before there exists a set $S_3'' \subset S_3'$ with card $S_3'' = m_3$ and

$$f(S_{3'}'') \cap (S_1'')_{i_1(1), i_2(1)}^{(3)} = 0, \qquad f(S_{3'}'') \cap (S_{2'}'')_{i_1(1)}^{(2)} = 0,$$

where $i_1^{(1)}$, $i_2^{(1)}$, $i_1^{(2)}$ are less than Ω_l. Again we construct $R(S_3'')$, and so on. Assume we have already completed this construction for all ordinals $\beta < \alpha < \Omega_r$. This means that there exists for all ordinals $\gamma < \alpha$ sets S_γ'', with card $S_\gamma'' = m_\gamma$ and so that for all $\delta < \gamma$ we have

$$f(S_{\gamma'}'') \cap (S_{\delta'}'')_{i_1(\delta), \ldots} = 0, \qquad (S_1'' = S_{1'}'')$$

where $i_1^{(\delta)} \cdots$ are ordinals less than Ω_l and $k = 1 + (\gamma - \delta)$. Now we construct the αth step as follows: Let a be any element of S_α''. Since card $(f(a)) < n$, there exists for every $\beta < \alpha$ an index $i^{(\beta)} < \Omega_l$ such that

$$f(a) \cap (S_{\beta'}'')_{i_1(\beta), \ldots, i_{k-1}(\beta)}^{(k)} = 0, \qquad k = 1 + (\alpha - \beta).$$

The number of possible choices for all the $i^{(\beta)}$, $\beta < \alpha$, does not exceed

$$n^{\aleph_r} < m_\alpha,$$

since the generalized continuum hypothesis was assumed to be true (it is clear from the generalized continuum hypothesis that $a^b \leqq \max(a^+, b^+)$ where a^+ is the cardinal number immediately following a). Thus since m_α is regular there exists a set $S_{\alpha'}''$ of power m_α such that for all $\beta < \alpha$

$$f(S_{\alpha'}'') \cap (S_{\beta'}'')_{i_1(\beta), \ldots, i_{k-1}(\beta)}^{(k)} = 0, \qquad k = 1 + (\alpha - \beta).$$

Now we construct our ramification system $R(S_{\alpha'}'')$. We continue this construction for every ordinal $\delta < \Omega_r$. Thus we obtain the sets

$$(S_{\delta'}'')_{i_1(\delta), \ldots}^{(\Omega_r)} = D_\delta, \qquad \delta = 1, 2, \cdots; \delta < \Omega_r. \tag{12}$$

The sets (12) all occur in the last (that is, Ω_rth) column of their ramification system. By construction they satisfy

$$f(D_{\delta_1}) \cap D_{\delta_2} = 0 \qquad \text{for } \delta_2 < \delta_1 < \Omega_r.$$

Thus we can put

$$D_\delta = S_\alpha'''$$

and this completes the proof of Theorem V.

In the first paper on this subject Lázár[8] proved (without assuming the continuum hypothesis) that Theorem V holds if $f(a)$ is finite and card $S=c$. In fact his proof gives that S is the union of $\aleph_0$ independent sets. One can ask the question what happens in the general case? It is easy to see that if $f(a)$ is finite and has not more than k elements and S is also finite, then S is the union of $2k+1$ independent sets, and $2k+1$ is best possible. By a method of König[9] it is easy to see that this also holds if S is countable. I conjectured that it is true for all sets S. Clearly this would be a consequence of the following result: Let G be a graph. Assume that all finite subgraphs of G are the union of r independent sets. (Two points are independent if they are not connected. A set of vertices is independent if any two of them are independent, and a graph G' is said to be the sum of r independent sets if the vertices of the graph can be split into r independent sets.) Then G is the union of r independent sets. De Bruijn recently proved this conjecture (written communication). In general perhaps the following result holds: Let card $(f(a))<n$. Then S is the sum of n independent sets (n is an infinite cardinal number).

Let S be any set of real numbers. We define S_k to be the set of all numbers

$$\sum_{r=1}^{k} c_r a_r, \qquad c_r \text{ rational}, \ a_r \in S.$$

Let H be a Hamel basis. It has been remarked[10] that the sets H_k cannot all be measurable. Indeed since $\bigcup_k H_k$ is the set of all real numbers, they cannot all have measure 0. Suppose that H_k has nonzero measure. If the inner measure of H_k would be positive, then by a well known theorem of Steinhaus[11] there would exist a δ such that if $|x|<\delta$, $x=a-b$, $a\in H_k$, $b\in H_k$. But then H_{2k} contains the set of all real numbers, which is clearly impossible.

Sierpinski[12] proved that there exist Hamel bases which are nonmeasurable and also Hamel bases of measure 0. We shall prove that for every k there exists a Hamel basis such that H_k has measure 0 but H_{k+1} is nonmeasurable.

Let $0\leqq A_1<A_2<\cdots$ be an infinite sequence of integers, such that every integer is the sum of $k+1$ A's, and the number of A's not ex-

[8] Compositio Math. vol. 3 (1936) p. 304.

[9] *Theorie der endlichen und unendlichen Graphen*, pp. 81–85.

[10] This remark is probably due to Sierpinski, but I do not remember for sure.

[11] Fund. Math. vol. 1, pp. 93–104.

[12] Ibid. vol. 1, pp. 105–111.

ceeding n is less than $c_1 n^{1/k+1}$. We shall prove the existence of such sequences later. Denote by B the set of real numbers in $(0, 1)$, which admit the representation:

$$\sum_{j=2}^{\infty} \frac{A_j}{j!}, \qquad \text{with } A_j < j \text{ an arbitrary term of } \{A_n\}.$$

First we prove that B_k has measure 0. It clearly suffices to prove that $\sum_{r=1}^{k} c_r b_r$ where the c's are fixed integers and $b_r \in B$ has measure 0. Let $x \in B_k$ be arbitrary. We have

$$x = \sum_{r=1}^{k} \sum_{j=2}^{\infty} c_r \frac{A_{i_j}(r)}{j!} = \sum_{j=2}^{\infty} \frac{u_j}{j!}, \qquad 0 \leqq A_{i_j}(r) < j,$$

where $|u_j| < c_2 \cdot j$ and we have, for u_j less than c', $j^{k/k+1}$ choices. Consider all the numbers

$$\sum_{k=2}^{n} \frac{u_k}{k!}.$$

The number of these numbers is less than $c'^n (n!)^{k/k+1}$. Consider all the intervals whose centers are all these points and whose length equals $4c_2/n!$. It follows from $|u_j| < c_2 j$ that all the $\sum_{k=2}^{\infty} u_k/k!$ are in the interior of these intervals, that is, the whole set $\sum_{r=1}^{k} c_k b_r$, $b_r \in B$, is covered by them. The sum of the length of these intervals is $< 4c(c'^n/n!^{1/k+1}) = o(1)$, which proves that B_k has measure 0.

Now we construct a Hamel basis $H \subset B$, with H_{k+1} nonmeasurable. First we construct a set $L \subset B$ such that L is rationally independent (that is, any finite subset of L is rationally independent) and L_{k+1} is nonmeasurable. Let $M = A_1 + A_2 + \cdots + A_{k+1}$. It is well known (and easy to see) that the real numbers z, $0 \leqq z \leqq 1$, in whose representation $z = \sum_{j=2}^{\infty} u_j/j!$, $0 \leqq u_j < j$, we have infinitely often $u_j = M$, have measure 1. Denote this set of real numbers by I_M. Let $\{F_\alpha\}$, $\alpha < \Omega_c$, be the collection of all the perfect subsets of I_M (Ω_c is the initial number of c). Suppose we have already constructed a set $L^{(\beta)}$ of power less than c, which is rationally independent and such that $L_{k+1}^{(\beta)}$ intersects all the F_α with $\alpha < \beta$. Let $z \notin F_\beta$, $z \notin \bigcup_j L_j^{(\beta)}$. Such a z exists since the power of $\bigcup_j L_j^{(\beta)}$ is less than c. Consider the equation

$$z = x_1 + x_2 + \cdots + x_{k+1}, \quad x_i \in B, \quad z = \sum_{j=2}^{\infty} \frac{u_j}{j!},$$

$$x_i = \sum_{j=2}^{\infty} \frac{A_j^{(i)}}{j!}, \qquad 0 \leqq u_j, A_j^{(i)} < j.$$

This equation is solvable. Also clearly we have c choices for x_1, c choices for x_2 after we have chosen x_1, and so on. Finally we have c choices for x_k after we have chosen $x_1, x_2, \cdots, x_{k-1}$. The proof of these statements is immediate, since $u_j = \sum_{i=1}^{k+1} A_j^{(i)}$ is solvable and since, in the representation of z, $M = A_1 + \cdots + A_{k+1}$ occurs infinitely often, say $u_{j_1} = u_{j_2} = \cdots = M$. We can clearly interchange the j_1th, j_2th, $\cdots$ digits of the x's (those digits are $A_1, A_2, \cdots, A_{k+1}$) and thus obtain the required c solutions. Thus we can clearly choose x_i, $i \leqq k$, so that the set consisting of $L^{(\beta)}$, the x_i, $i \leqq k$, and z should be rationally independent. But then the set consisting of $L^{(\beta)}$ and the x_i, $i \leqq k+1$, is also rationally independent. This set we denote by $L^{(\beta+1)}$. Clearly $L_{k+1}^{(\beta+1)}$ intersects F_β, and the power of $L^{(\beta+1)}$ is less than c. Put now $L = \bigcup_{\beta<\omega_c} L^{(\beta)}$. Clearly L_{k+1} intersects all perfect subsets of I_M, thus can not have measure 0 (I_M has measure 1), but since L is rationally independent, L_{k+1} can not have positive inner measure (as was shown before). Thus L_{k+1} is nonmeasurable. Now since B_{k+1} clearly consists of all real numbers, we can find a Hamel basis $H \subset B$ which contains L. Clearly H_{k+1} is also nonmeasurable and $H_k \subset B_k$ has measure 0, which completes the proof.

We now have to construct our sequence A with the required properties: Denote by T_r, $0 \leqq r < k+1$, the sequence of integers of the form $\sum_{j=0}^{n} \epsilon_j 2^j$, $n = 1, 2, \cdots$, $\epsilon_j = 0$ or 1, and all the ϵ_j are 0 except possibly for those with $j \equiv r \pmod{(k+1)}$. Consider now the sequence $\bigcup_r T_r$, this sequence clearly has the required properties.[13]

Let S be a set of power greater than $\aleph_0$ but smaller then the first strongly inaccessible cardinal number ($>\aleph_0$). It is well known that no countably additive two-valued measure can exist for the subsets of S, so that elements have measure 0 and the whole set S has measure 1. Ulam now raised the question (oral communication): What is the smallest cardinal number n so that there should exist n two-valued measures defined for the subsets of S (elements of S having measure 0 and S having measure 1, in each of them) with the property that each subset of S is measurable in at least one of these measures? Ulam proved that $n \geqq \aleph_0$. Alaoglu and I proved $n \geqq \aleph_1$. I now present our proof.

Suppose the result is false. Then there exist countably many measures $M_1, M_2, \cdots$ so that each subset of S is measurable for at least one of these measures. Split the set S into the union of $\aleph_1$ sets each nonmeasurable in M_1.[14] Clearly only countably many can have positive measure in any M_k, thus there must exist two of them, S_1 and

[13] This example is due to Stöhr, Math. Zeit. vol. 42, pp. 739–743.

[14] S. M. Ulam, Fund. Math. vol. 16 (1930) p. 142.

S_1' say, which are both nonmeasurable in M_1 and whose union does not have positive measure in any M_k. Split now $S-S_1-S_1'$ into the union of $\aleph_1$ disjoint sets each nonmeasurable in M_2. Denote these sets by $U_1, U_2, \cdots, U_\xi, \cdots$. It is easy to see that among the sets $S_1+S_1'+U_\xi$ there are only countably many of positive measure in any M_k. Thus there exist two of them, S_2 and S_2' say, such that $S_1+S_1'+S_2+S_2'$ does not have positive measure for any M_k. Consider now the set $S-S_1-S_1'-S_2-S_2'$, and repeat the same operation. We can clearly repeat this operation for any k, and the set $\bigcup_i S_i$ will be nonmeasurable for any M_k, since it contains S_k but is disjoint to S_k'. We cannot decide the question whether $n>\aleph_1$. It seems probable that n is greater than the power of S.

Some simple remarks about ordinal numbers: (1) A sequence of ordinals (not necessarily countable) $\beta_i<\Omega_k$ is said to be rarified if $\lim \beta_{i+1}-\beta_i=\Omega_k$, also every bounded sequence is said to be rarified (that is, a sequence for which $\lim \beta_i=\Omega_k$). Sierpinski[15] remarked that the ordinals $\alpha<\Omega_1$ are the sum of $\aleph_0$ but not of a finite number of rarified sequences. Assume that Ω_k has an immediate predecessor. Then the ordinals $\alpha<\Omega_k$ are the sum of Ω_{k-1} but not fewer rarified sequences. If Ω_k does not have an immediate predecessor and Ω_l is the smallest ordinal cofinal with it, then the ordinals $\alpha<\Omega_k$ are the sum of Ω_l but not fewer rarified sequences. If Ω_k is weakly inaccessible, that is, $\Omega_k=\Omega_l$, then the ordinals $\alpha<\Omega_k$ are not the sum of fewer than Ω_k rarified sequences, or the weakly inaccessible numbers are the only ones which are not the sum of fewer rarified sequences.

Similarly we can define a sequence $\beta_i<\Omega_k$ to be r-rarified if $\lim \beta_{i+1}-\beta_i=\Omega_r$. Then the ordinals $\alpha<\Omega_k$ are the sum of $\min(\Omega_r, \Omega_l)$ but not fewer r-rarified sequences.

(2) Dushnik[16] proved the following theorem: Let m be a regular number, Ω_k the initial number belonging to it. Let $f(\alpha)$, $\alpha<\Omega_k$, be such that for all α, $f(\alpha)<\alpha$. Then there exists an ordinal β such that the equation $f(\alpha)=\beta$ has m solutions. For singular numbers the theorem is false. The following generalization holds: Assume that Ω_k is not cofinal to ω. Then there exists an ordinal β and a sequence α_i cofinal with Ω_k such that $f(\alpha_i)\leqq\beta$. For regular numbers one immediately obtains from this Dushnik's theorem. If Ω_k is cofinal to ω, the analogous result clearly does not hold. Dushnik's proof would easily give a proof of this theorem (oral communication) but perhaps the following proof is of some interest: Suppose the theorem is false. Then to every β there exists a $\phi(\beta)<\Omega_k$ such that for $\delta\geqq\phi(\beta)$, $f(\delta)>\beta$.

[15] Revista de Sciencias (Lima) vol. 41 (1939) pp. 289–296.

[16] Bull. Amer. Math. Soc. vol. 37 (1931) pp. 860–862.

Consider now

$$\gamma = \phi(1) + \phi(\phi(1)) + \cdots .$$

Clearly $\gamma < \Omega_k$ since Ω_k is not cofinal with ω. But then from the definition of γ, $f(\gamma) \geqq \gamma$, an evident contradiction; this completes the proof.

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ON A PROBLEM OF G. BIRKHOFF

TADASI NAKAYAMA AND JUNJI HASHIMOTO

In his book *Lattice theory*, G. Birkhoff proposed to prove that the representation of a finite partially ordered system as the product of indecomposable factors is unique within pairwise isomorphism of factors.[1] The present short note is to show that this is not the case in general. A simple counterexample, and indeed one of the simplest, perhaps, can be constructed as follows:

Let X be the lattice $\{0, 1\}$ of two elements $0, 1$ $(0<1)$, for instance, and A be the partially ordered system

$$I + X + X^2 + X^3 + X^4 + X^5,$$

where I resp. X^i stands for the one-lattice resp. the direct product of i copies of X, and where $+$ means direct summation. The finite partially ordered system A may be expressed also by $f(X)$ with the polynomial $f(x) = 1+x+x^2+x^3+x^4+x^5$. Since every X^i has the up to isomorphism unique decomposition into indecomposable factors, $X^i = XX \cdots X$ (i factors), one sees easily that direct decompositions of A are, in the sense of isomorphism, in 1-1 correspondence with factorizations of polynomial $f(x) = 1+x+x^2+x^3+x^4+x^5$ into factors with non-negative rational integral coefficients. But our $f(x)$ has two distinct decompositions into factors which are irreducible in the prescribed sense, namely

$$f(x) = (1 + x)(1 + x^2 + x^4) = (1 + x^3)(1 + x + x^2).$$

Two direct decompositions

$$A = (I + X)(I + X^2 + X^4) = (I + X^3)(I + X + X^2)$$

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[1] G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Publications, vol. 25, New York, 1940, Problem 8.

of A give thus a counter-example to the conjecture.

Nevertheless the conjecture is true in the important special case of directly sum-indecomposable systems, as will be shown in a forthcoming paper by one of the writers.[2] Making use of the result we may formulate a criterion of unique decomposition as follows: First we decompose the given system into the direct sum of sum-indecomposable systems, then decompose the so-obtained summands into a product of (product-)indecomposable factors, construct a polynomial by taking a variable for each isomorphism type of such (product-)indecomposable factors, and finally examine whether the factorization of the polynomial (with several variables, perhaps) into factors irreducible within the domain of polynomials with non-negative rational integral coefficients is unique.[3]

We want also to suggest a generalization of the notion of partially ordered system by allowing negative coefficients, as in the case of algebraic complex in topology. This change of notion will make Birkhoff's conjecture true.

Added in proof: Our above example, communicated to Professor Birkhoff, has been incorporated into the revised second edition of *Lattice theory*, p. 26.

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[2] J. Hashimoto, *On the product decomposition of partially ordered set*, Mat. Japonicae vol. 1 (1949).

[3] This clarifies two extreme cases cited by Birkhoff, loc. cit.

THE DIFFERENCE BETWEEN CONSECUTIVE PRIME NUMBERS. IV

R. A. RANKIN

1. **Introduction.** Let p_n denote the nth prime, and let Θ be the lower bound of all positive numbers σ_0 such that no Dirichlet L-function $L(s, \chi)$ has a zero at $s=\sigma+it$, where $\sigma>\sigma_0$. Write

$$l = \liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n}.$$

The purpose of this paper is to combine the methods used in two earlier papers[1] in order to prove the following theorem.

THEOREM.

$$l \leqq c(1 + 4\Theta)/5, \tag{1}$$

where $c<42/43$.

The precise definition of c is given by (18) and (8) below.

The theorem is an improvement on the result $l\leqq(1+4\Theta)/5$ obtained in II. It was shown in III that

$$l \leqq d = 0.966\cdots < 57/59. \tag{2}$$

The numbers c and d are connected by the relation

$$1 - d = (3 + 2^{3/2})(1 - c)/4 > 1 - c,$$

so that (1) is an improvement on (2) only if Θ is not too close to unity, in fact, if $\Theta<0.986\cdots$. In particular, if the "grand Riemann hypothesis" is true, that is, if $\Theta=1/2$, we have (using the value of c given by (18))

$$l \leqq 3c/5 < 109/186 = 0.58602\cdots.$$

2. **Notation.** Since (2) is sharper than (1) for $\Theta=1$, we shall assume that $\Theta<1$, and write

$$\sigma = (1 + 4\Theta)/5. \tag{3}$$

Let N be a large positive integer, and define

$$X = N^{1-\sigma}(\log N)^{-8(1+\gamma)} \tag{4}$$

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[1] See [2] and [3]; I refer to these as II and III respectively. Numbers in brackets refer to the references cited at the end of the paper.

as done in equation (6.7) of II; here γ is a positive constant which may be taken to be 2.

Let $\tilde{\omega}(\nu)$ be the number of prime pairs p, p' which satisfy

$$N < p' \leqq p \leqq 2N, \qquad p - p' = \nu,$$

where ν is an integer. We suppose that h, k, m, n and r are integers such that, for some positive A_1, A_2 independent of N,

$$A_1 \log N < 2k/3 < h < k < A_2 \log N, \tag{5}$$

$$0 \leqq m < n < A_2 \log N, \tag{6}$$

$$r = k - h.$$

The letters a, c, α, β denote positive numbers independent of N, and we write

$$D = \prod_{p>2}\left(1 + \frac{1}{p(p-2)}\right), \tag{7}$$

and take

$$\lambda = 7.3566 > e^{-2\gamma}\Lambda_0/2, \tag{8}$$

where γ is Euler's constant and Λ_0 is Buchstab's number 46.67347 (see relation (4) of III).

For $\mu = 1, 2, \cdots, k$ we define a_μ as follows:

$$a_\mu = \begin{cases} \alpha & (0 < \mu \leqq r,\ k - r < \mu \leqq k), \\ \beta & (r < \mu \leqq k - r), \end{cases} \tag{9}$$

and write

$$\Psi(\theta) = \sum_{\mu=1}^{k} a_\mu e^{4\pi i\mu\theta}. \tag{10}$$

Then

$$|\Psi(\theta)|^2 = \xi(0) + 2\sum_{\nu=1}^{k-1} \xi(\nu) \cos 4\pi\nu\theta, \tag{11}$$

where

$$\xi(\nu) = \sum_{i-j=\nu,\ 1\leqq j\leqq i\leqq k} a_i a_j.$$

3. Lemmas.

LEMMA 1. *We have*

$$\xi(\nu) = \begin{cases} 2(r-\nu)\alpha^2 + 2\nu\alpha\beta + (k-2r-\nu)\beta^2 & (0 \leqq \nu < r), \\ 2r\alpha\beta + (k-2r-\nu)\beta^2 & (r \leqq \nu < k-2r), \\ (2r-k+\nu)\alpha^2 + 2(k-r-\nu)\alpha\beta & (k-2r \leqq \nu < k-r), \\ (k-\nu)\alpha^2 & (k-r \leqq \nu < k), \end{cases}$$

and

$$\sum_{\nu=1}^{k-1} \xi(\nu) = \frac{1}{2}\{2r\alpha + (k-2r)\beta\}^2 + O(\log N).$$

This follows easily from (9) and (11), and we omit the proof. It is in this lemma that the restriction $k-r=h>2k/3$ made in (5) is used.

LEMMA 2.

$$\sum_{1 \leqq q \leqq X} \frac{\mu^2(q)}{\phi^2(q)} \sum_{f=0,(f,q)=1}^{q-1} e^{4\pi i \nu f/q} = \frac{2}{D} \prod_{p \mid \nu, p>2} \left(\frac{p-1}{p-2}\right) + O(\nu X^{-1}(\log\log X)^2).$$

Here $\mu(q)$ is the Möbius function, $\phi(q)$ is Euler's function, and D is defined by (7).

It follows from the well known relation

$$\sum_{f=0,(f,q)=1}^{q-1} e^{4\pi i \nu f/q} = \sum_{d \mid (2\nu,q)} d\mu\left(\frac{q}{d}\right)$$

that

$$\begin{aligned} \sum_{1 \leqq q \leqq X} \frac{\mu^2(q)}{\phi^2(q)} \sum_{f=0,(f,q)=1}^{q-1} e^{4\pi i \nu f/q} &= \sum_{q \leqq X} \frac{\mu(q)}{\phi^2(q)} \prod_{p \mid (2\nu,q)} (1-p) \\ &= \sum_{q=1}^{\infty} \frac{\mu(q)}{\phi^2(q)} \prod_{p \mid (2\nu,q)} (1-p) \\ &\quad - \sum_{q>X} \frac{\mu(q)}{\phi^2(q)} \prod_{p \mid (2\nu,q)} (1-p). \end{aligned}$$

By a straightforward reduction the first term on the right may be shown to equal

$$\frac{2}{D} \prod_{p \mid \nu, p>2} \left(\frac{p-1}{p-2}\right).$$

See, for example, §§3.21–3.23 of [1]. Since $1/\phi(n) = O(n^{-1} \log\log n)$, the second term on the right is

$$O\left\{X^{-1}(\log\log X)^2 \prod_{p|2\nu}(p-1)\right\} = O\{\nu X^{-1}(\log\log X)^2\}.$$

LEMMA 3. *If $s \geqq 0$ and is independent of N, then*

$$\sum_{\nu=m}^{n-1} \nu^s \prod_{p|\nu, p>2}\left(\frac{p-1}{p-2}\right) = D\sum_{\nu=m}^{n-1}\nu^s + O\{(\log N)^s(\log\log N)^2\}.$$

The proof is similar to that of the lemma in III. We suppose that d is any product $p_{i_1}p_{i_2}\cdots p_{i_j}$ of distinct odd primes, and let d' be the associated product

$$d' = (p_{i_1}-2)(p_{i_2}-2)\cdots(p_{i_j}-2).$$

We permit d to take the value unity and then put $d'=1$. Then, clearly,

$$\sum_{d<n}\frac{1}{dd'} = \sum_{d\geqq 1}\frac{1}{dd'} + O\left\{\frac{1}{n}(\log\log 3n)^2\right\}$$

$$= D + O\left\{\frac{1}{n}(\log\log 3n)^2\right\},$$

since $1/d' = O\{d^{-1}(\log\log d)^2\}$ for large[2] d; and

$$\prod_{p|\nu, p>2}\left(\frac{p-1}{p-2}\right) = \prod_{p|\nu, p>2}\left(1+\frac{1}{p-2}\right)$$

$$= \sum_{d|\nu}\frac{1}{d'}.$$

For a given d', ν is a multiple of d, say $\nu = \mu d$, and, since $m \leqq \nu < n$, μ takes all integer values in the interval $m/d \leqq \mu < n/d$, which we denote by $I(d)$. Clearly

$$\sum_{\mu\in I(d)}(\mu d)^s = \frac{n^{s+1}-m^{s+1}}{(s+1)d} + O(n^s),$$

and

$$\sum_{\nu=m}^{n-1}\nu^s = \frac{n^{s+1}-m^{s+1}}{s+1} + O(n^s).$$

[2] It is possible to dispense with the factor $(\log\log 3n)^2$.

Hence

$$\sum_{\nu=m}^{n-1} \nu^s \prod_{p\mid\nu,\,p>2}\left(\frac{p-1}{p-2}\right) = \sum_{d<n}\frac{1}{d^s}\sum_{\mu\in I(d)}(\mu d)^s$$

$$= \frac{n^{s+1}-m^{s+1}}{s+1}\sum_{d<n}\frac{1}{dd^s} + O\left(n^s\sum_{d<n}\frac{1}{d^s}\right)$$

$$= \frac{n^{s+1}-m^{s+1}}{s+1}\left\{D + O\left(\frac{1}{n}(\log\log 3n)^2\right)\right\}$$

$$+ O\{n^s(\log\log 3n)^2\log 2n\}$$

$$= D\sum_{\nu=m}^{n-1}\nu^s + O\{n^s(\log 2n)^2\},$$

and the lemma follows from (6).

LEMMA 4.

$$\sum_{1\le q\le X}\frac{\mu^2(q)}{\phi^2(q)}\sum_{f=0,(f,q)=1}^{q-1}\sum_{\nu=1}^{k-1}\xi(\nu)e^{4\pi i\nu f/q}$$

$$= 2\sum_{\nu=1}^{k-1}\xi(\nu) + O\{\log N(\log\log N)^2\}$$

$$= \{2r\alpha + (k-2r)\beta\}^2 + O\{\log N(\log\log N)^2\}.$$

By Lemma 2, the left-hand member is equal to

$$\frac{2}{D}\sum_{\nu=1}^{k-1}\xi(\nu)\prod_{p\mid\nu,\,p>2}\left(\frac{p-1}{p-2}\right) + O\left\{kX^{-1}(\log\log X)^2\sum_{\nu=1}^{k-1}\xi(\nu)\right\}.$$

It follows from Lemmas 1 and 3 (with $s=0, 1$) that this equals

$$2\sum_{\nu=1}^{k-1}\xi(\nu) + O\{\log N(\log\log N)^2\} + O\{X^{-1}(\log N)^2(\log\log X)^2\},$$

and the lemma follows from (4) and the second part of Lemma 1.

4. **Proof of the theorem.** By considering the expression

$$I(N, k) = \int_0^1 |S(\theta)\Psi(\theta)|^2 d\theta,$$

where

$$S(\theta) = \sum_{N<p\le 2N} e^{2\pi i\theta p}\log p \qquad (p \text{ prime}),$$

and $\Psi(\theta)$ is defined by (10), it is possible to show that, for large N,

$$
\begin{aligned}
I(N, k) &= \xi(0)N \log N\{1 + o(1)\} \\
&\quad + 2\{1 + o(1)\} \log^2 N \sum_{\nu=1}^{k-1} \xi(\nu)\bar{\omega}(2\nu) \\
&> N\{1 + o(1)\} \sum_{1 \le q \le X} \frac{\mu^2(q)}{\phi^2(q)} \sum_{f=0,(f,q)=1}^{q-1} \left|\Psi\left(\frac{f}{q}\right)\right|^2 \\
&\quad + o(N \log^2 N) \qquad (12) \\
&= \xi(0)N\{1 + o(1)\} \sum_{1 \le q \le X} \frac{\mu^2(q)}{\phi(q)} \\
&\quad + 2N\{1 + o(1)\} \Re \sum_{1 \le q \le X} \frac{\mu^2(q)}{\phi^2(q)} \\
&\quad \cdot \sum_{f=0,(f,q)=1}^{q-1} \sum_{\nu=1}^{k-1} \xi(\nu) e^{4\pi i \nu f/q} + o(N \log^2 N).
\end{aligned}
$$

The proof of this is similar in every respect to the proof of (4.1) and (6.8) of II, except that $\Psi(\xi)$ is defined by (10) and is not $\sum_{\nu=1}^{k-1} e^{2\pi i \nu \theta}$ as in II. The restriction $k < \log N$ made in II is unessential and can be replaced by $k < A_2 \log N$.

It follows from (3), (4), (12), Lemma 4, and Lemma 4 of[3] II that

$$
\begin{aligned}
&2\{1 + o(1)\} \log^2 N \sum_{\nu=1}^{k-1} \xi(\nu)\bar{\omega}(2\nu) \\
&\quad > 4N\{1 + o(1)\} \sum_{\nu=1}^{k-1} \xi(\nu) - \sigma N \log N\{1 + o(1)\}\xi(0) \\
&\quad\quad + o(N \log^2 N) \qquad (13) \\
&\quad = 2N\{2r\alpha + (k - 2r)\beta\}^2\{1 + o(1)\} \\
&\quad\quad - \sigma N \log N\{2r\alpha^2 + (k - 2r)\beta^2\}\{1 + o(1)\} \\
&\quad\quad + o(N \log^2 N).
\end{aligned}
$$

Now, just as in III, it can be shown by Buchstab's form of Brun's method[4] that, for sufficiently large N,

[3] There is a misprint in the enunciation of the lemma which should read: "*If X is large* $\sum_{1 \le q \le X} \mu^2(q)/\phi(q) \sim \log X$."

[4] We must consider the interval $(N, 2N)$ instead of the interval $(x/\log x, x)$. Our only reason for considering an interval of the form $(N, 2N)$ is that it enables us to quote results directly from II. As far as Brun's method is concerned, the inequality stated remains true if we replace the left-hand side $\bar{\omega}(2\nu)$ by the number of prime pairs p, p' which satisfy $AN < p' \le p \le (A+1)N$, $p - p' = 2\nu > 0$ where $A \ge 0$.

$$\bar{\omega}(2\nu) < \frac{4\lambda N}{D(\log N)^2} \prod_{p\mid\nu, p>2} \left(\frac{p-1}{p-2}\right),$$

where λ is defined by (8). It follows from Lemmas 1 and 3 that

$$\begin{aligned}\sum_{\nu=k-r}^{k-1} \xi(\nu)\bar{\omega}(2r) &< \frac{4\lambda N}{(\log N)^2} \sum_{\nu=k-r}^{k-1} \xi(\nu) + O\left\{\frac{N}{\log N}(\log\log N)^2\right\} \\ &= \frac{2\lambda\alpha^2 r^2 N}{(\log N)^2} + O\left\{\frac{N}{\log N}(\log\log N)^2\right\}\end{aligned} \tag{14}$$

since $r = O(\log N)$.

Accordingly, if we make the assumption that $\bar{\omega}(2\nu) = 0$ for all $\nu < h = k - r$ we have, from (13) and (14), that

$$\begin{aligned}&\left\{\frac{2r\alpha + (k-2r)\beta}{\log N}\right\}^2 \{1 + o(1)\} \\ &\quad - \frac{1}{2}\sigma\left\{\frac{2r\alpha^2 + (k-2r)\beta^2}{\log N}\right\}\{1 + o(1)\} \\ &\quad < 2\lambda\alpha^2\left(\frac{r}{\log N}\right)^2\{1 + o(1)\}.\end{aligned} \tag{15}$$

We now take $\alpha = 1$, $\beta = 2^{1/2}$, and put

$$\begin{aligned}k &= \left[\frac{1}{2}\sigma\left(1 + \frac{2^{1/2}a}{\lambda}\right)\log N\right], \\ h &= \left[\frac{1}{2}\sigma\left(1 - \frac{a}{\lambda}\right)\log N\right],\end{aligned} \tag{16}$$

where a is a fixed positive number satisfying

$$a < 3 - 2^{3/2}. \tag{17}$$

It is easily verified that, with this choice of a, $h > 2k/3$.

Then, for large N,

$$r = \frac{a\sigma}{2\lambda}(2^{1/2} + 1)\log N + O(1),$$

$$2r\alpha + (k-2r)\beta = (2 - 2^{1/2})(k + 2^{1/2}h) = \frac{1}{2^{1/2}}\sigma\log N + O(1),$$

$$2r\alpha^2 + (k-2r)\beta^2 = 2h = \sigma\left(1 - \frac{a}{\lambda}\right)\log N + O(1).$$

It follows from (15) that

$$\frac{1}{2}\sigma^2 - \frac{1}{2}\sigma^2\left(1 - \frac{a}{\lambda}\right) \leqq 2\lambda\left\{\frac{\sigma a(2^{1/2}+1)}{2\lambda}\right\}^2,$$

that is, that

$$a \geqq 3 - 2^{3/2},$$

which contradicts (17).

We conclude, accordingly, that $\bar{\omega}(2\nu) > 0$ for some integer ν satisfying

$$\nu < \frac{1}{2}\,\sigma\left(1 - \frac{a}{\lambda}\right)\log N.$$

It follows from this inequality, and from (17), that

$$l \leqq \sigma c$$

where

$$c = 1 - (3 - 2^{3/2})/\lambda = 0.97667\cdots < \frac{42}{43}, \tag{18}$$

and this proves the theorem.

Note added 24 *April* 1949. Professor Erdös has informed me that considerable improvements have recently been made in Brun's method by A. Selberg and others; this will presumably mean that Buchstab's constant Λ_0, and consequently my constants c and d (see equations (2) and (18)), can be replaced by smaller values.

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FAMILIES OF SOLUTIONS OF THE UNILATERAL MATRIX EQUATION

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1. Introduction and summary. Let $\mathfrak{F}$ be a field of characteristic zero, and let R_m $(m=0, 1, \cdots, r)$ be $n\times n$ matrices with elements in $\mathfrak{F}$. M. H. Ingraham [1][1] has shown that the unilateral matrix equation $\sum_{m=0}^{r} R_m X^m = 0$ may have an infinitude of solutions. In this paper a study is made of the conditions under which such an infinitude of solutions will exist. All matrices considered in this paper will be square and of order n unless otherwise specified.

The matrix X_1 is *similar* to the matrix X_2 if there exists a nonsingular matrix T such that $TX_1T^{-1}=X_2$ and the elements of X_1, X_2 and T are in $\mathfrak{F}$. The *polynomial domain* $\mathfrak{F}[\lambda]$ of $\mathfrak{F}$ is a *principal ideal ring*. The matrix $\sum_{m=0}^{r} R_m\lambda^m$ is a λ-*matrix*, that is, its elements are in $\mathfrak{F}[\lambda]$. A square λ-matrix is *unimodular* if it has an inverse which is also a λ-matrix. Two λ-matrices A and B are *left associates* if there is a unimodular matrix U such that $UA=B$. The relationship of *left associativity* is an *equals relationship*. Every square λ-matrix is a left associate of a unique matrix which is in *canonical triangular form*. This matrix has all elements below the main diagonal equal to zero. If a diagonal element is zero, its row consists entirely of zeros. The leading coefficient of each nonzero diagonal element is unity, and the elements in that column are reduced modulo the diagonal element.[2]

If M, P, and A are λ-matrices such that $M=PA$, then A is said to be a right divisor of M, and M is a left multiple of A. Every pair of λ-matrices has a *greatest common right divisor* (*g.c.r.d.*) D and every pair of nonsingular λ-matrices has a *least common left multiple* (*l.c.l.m.*) M which is unique up to left associates [2].

The question arises: Given two nonsingular matrices A and B, what other right divisors of their l.c.l.m. M are there which are not left associates of A or B? It has been shown[3] that X_1 is a solution

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] For these definitions and terms see MacDuffee [2]. The Hermite normal form as defined by MacDuffee is the form generally used. The canonical triangular form defined here is used in keeping with that used by Ingraham [1]. It is obtained in the same manner as the Hermite normal form except the operations are carried out in the columns in reverse order.

[3] The proof of the factor theorem for polynomials with coefficients in a ring with an identity element is given by Albert [3].

of $\sum_{m=0}^{r} R_m X^m = 0$ if and only if $\lambda I - X_1$ is a right divisor of $\sum_{m=0}^{r} R_m \lambda^m$. Therefore, if there exist two distinct solutions X_1 and X_2, the l.c.l.m. of $\lambda I - X_1$ and $\lambda I - X_2$ is a right divisor of $\sum_{m=0}^{r} R_m \lambda^m$. If this l.c.l.m. has a right divisor of the form $\lambda I - X_3$, where X_3 is different from X_1 and X_2, then X_3 is another solution of the unilateral equation. Thus, a study of the above problem yields results applicable to the unilateral matrix equation. The λ-matrix A is a right divisor of the λ-matrix M if and only if its canonical triangular form is a right divisor of the canonical triangular form of M. Consequently, the matrices A, B, and M will be considered as being in canonical triangular form.

If the λ-matrix $R(\lambda)$ has a set of right divisors whose canonical triangular forms have the same main diagonal, then the set will be called a *family of right divisors* of $R(\lambda)$. A matrix of the form $\lambda I - X$ where X has elements in $\mathfrak{F}$ will be called a *monic matrix of degree one.* It is easily shown that two monic matrices are left associates if and only if they are identical. If a family of right divisors of $R(\lambda)$ are the left associates of monic matrices of degree one, the matrices X obtained from the monic matrices are solutions of $R(X) = 0$ and will be called a *family of solutions of* $R(X) = 0$.

It is shown that a necessary and sufficient condition for the existence of an infinite family of solutions of the unilateral matrix equation is that there be two distinct solutions which are similar. All matrices similar to a given solution are solutions of the equation if and only if their minimal polynomial is a common divisor of the elements of $\sum_{m=0}^{r} R_m \lambda^m$. An interesting result arises as a by-product of this second result, namely: The equation $\sum_{m=0}^{r} R_m \cdot \times X^m = 0$ ($\sum_{m=0}^{r} X^m \cdot \times R_m = 0$), where $R_m \cdot \times X^m$ is the direct product of R_m and X^m [2], has a solution if and only if the elements of $\sum_{m=0}^{r} R_m \lambda^m$ have a greatest common divisor which is not a unit. (In this problem the order of X is not restricted to n.) It is also shown that if $\sum_{m=0}^{r} R_m X^m = 0$ has a solution, and if the determinant of $\sum_{m=0}^{r} R_m \lambda^m$ is zero identically, then there is an infinitude of solutions of the equation.

2. **Families of solutions.** The following theorem provides a basis for the results obtained in this paper.

THEOREM 1. *If $P = QA$, where P is a nonsingular λ-matrix, P and A being in canonical triangular form, and if for some values j and k, where $j > k$, it is true that q_{kk} and a_{jj} are not relatively prime; then there exists a family of right divisors of P which are not left associates. Furthermore, if A is the left associate of a monic matrix of degree one, then the mem-*

bers of the family, with a finite number of exceptions, are also left associates of monic matrices of degree one.

Since P and A are triangular matrices and P is nonsingular, it follows that Q and A are nonsingular, and Q is also triangular. Hence, $p_{mm}=q_{mm}a_{mm}$ $(m=1, 2, \cdots, n)$.

Consider a possible factorization of P into $P=BA'$, where A' has the same diagonal elements as A and is in canonical triangular form. One such factorization exists, namely $P=QA$. The matrix B will be a triangular matrix.

Represent the greatest common divisor of two polynomials $f(\lambda)$ and $g(\lambda)$ by the symbol $(f(\lambda), g(\lambda))$.

Select the values of k and j so that: $(q_{kk}, a_{jj})\neq 1$ and $j>k$; $(q_{mm}, a_{pp})=1$ when $m<p$ and $p>j$; $(q_{mm}, a_{jj})=1$ for $m<k$. Also let $a'_{mp}=a_{mp}$ $(m,\ p=1,\ 2, \cdots, j-1;\ p=j,\ m=k+1,\ k+2, \cdots, j-1)$. Since $b_{mm}=q_{mm}$ $(m=1, 2, \cdots, n)$ and $b_{mp}=q_{mp}$ $(m,\ p=1, 2, \cdots, j-1;\ p=j,\ m=k+1, k+2, \cdots, j-1)$, the element a'_{kj} is obtained from the following equation:

$$p_{kj}=q_{kk}a'_{kj}+\sum_{m=k+1}^{j-1}q_{km}a_{mj}+b_{kj}a_{jj}$$

that is,

$$q_{kk}a'_{kj}\equiv p_{kj}-\sum_{m=k+1}^{j-1}q_{km}a_{mj} \bmod a_{jj}.$$

This congruence has a solution, namely a_{kj}. Therefore (q_{kk}, a_{jj}) divides $p_{kj}-\sum_{m=k+1}^{j-1}q_{km}a_{mj}$, and $a'_{kj}=x_1(\lambda)+(\sum_{m=1}^{p}a_m\lambda^{p-m})f_1(\lambda)$ where $x_1(\lambda)$ and $f_1(\lambda)$ are uniquely determined, p is the degree of (q_{kk}, a_{jj}) and a_m $(m=1, 2, \cdots, p)$ are arbitrary parameters. The remaining elements of A' are uniquely determined in terms of a'_{kj} and the other elements previously found, since $(q_{mm}, a_{jj})=1$ $(m<k)$, and $(q_{mm}, a_{pp})=1$ $(p>j,\ p>m)$.

Therefore, in solving for A' an infinite family of right divisors of P is obtained. Furthermore, each member of the family is in canonical triangular form. Hence, no two distinct members of the family may be left associates.

The existence of a family of right divisors of P has been shown. In the construction above the parameters first occur in a'_{kj} and may occur in the elements a'_{mp} $(m<k,\ p=j;\ p=j+1, \cdots, n)$. When the parameters occur in an element, they will appear in the coefficients of the various powers of λ in that element. In this case the coefficients will be polynomials in the parameters. A certain combination of values

for the parameters will yield A from A'. Assign these values to all but one of the parameters. The modified matrix A' will involve only one parameter, and by assigning the proper value to this parameter A will be obtained from A'.

If there is a unimodular matrix T such that $TA=\lambda I-X$, then it is true that A' will be the left associate of a matrix $\lambda I-X'$ except possibly for a finite set of values of the parameter. The necessary and sufficient conditions that the canonical triangular matrix $A=\sum_{m=0}^{s}A_m\lambda^m$ be the left associate of a monic matrix of degree one are: The degree of $\prod_{m=1}^{n}a_{mm}$ should equal n, the degree of $\prod_{m=1}^{j}a_{mm}$ should be equal to or less than j $(j<n)$, and the matrix $W_1=\|A_sA_{s-1}\cdots A_1\|$ shall be of rank n. Since $a'_{mm}=a_{mm}$ $(m=1,2,\cdots,n)$ the first conditions are satisfied. The matrix W_1 consists of at most n nonzero columns.[4] That is, the rank of W_1 is the rank of an $n\times n$ matrix D formed from the nonzero columns of W_1. Because $|D|$ is a polynomial in the parameter and $|D|\not\equiv 0$ (since $TA=\lambda I-X$), it follows that $|D|=0$ for only a finite set of values of the parameter. Therefore there is an infinite family of right divisors of P which are left associates of monic matrices of degree one.

The following theorem is due to V. J. Varineau [4].

THEOREM 2. *If the least common left multiple of two nonsingular matrices A and B, with elements in a principal ideal ring, is M, and their greatest common right divisor is D; then $|M|\cdot|D|=u|A|\cdot|B|$ where u is a unit.*

THEOREM 3. *The unilateral matrix equation $\sum_{m=0}^{r}R_mX^m=0$ has an infinite family of solutions if and only if there exist two distinct solutions X_1 and X_2 which are similar.*

Let A, B, and M be the canonical triangular forms of $\lambda I-X_1$, $\lambda I-X_2$, and the l.c.l.m. of $\lambda I-X_1$ and $\lambda I-X_2$ respectively. Since X_1 and X_2 are solutions of $\sum_{m=0}^{r}R_mX^m=0$, A and B are right divisors of $\sum_{m=0}^{r}R_m\lambda^m$. Therefore M is a right divisor of $\sum_{m=0}^{r}R_m\lambda^m$. It will be shown that M has a family of right divisors of the form $\lambda I-X'$, and thus $\sum_{m=0}^{r}R_mX^m=0$ has a family of solutions.

Since $|A|=|B|=|\lambda I-X_1|=|\lambda I-X_2|$, M is nonsingular. Also $M=PA=QB$, where P and Q are triangular λ-matrices. Either $a_{mm}=b_{mm}$ $(m=1,2,\cdots,n)$ or else not all $a_{mm}=b_{mm}$ $(m=1,2,\cdots,n)$. If $a_{mm}=b_{mm}$ $(m=1,2,\cdots,n)$ it follows that $(p_{kk},a_{jj})\neq 1$ for some values j and k where $j>k$. If this were not so, there would be a

[4] Published as note without proof in Ingraham [1]. Proof given in a paper by Bell [5].

unique right divisor of M having the elements a_{mm} as the main diagonal elements, whereas there are at least two such divisors, namely A and B. The theorem follows from Theorem 1.

Assume, in the second case, that not every a_{mm} is equal to the corresponding b_{mm}. Let j be the least value of m such that $a_{jj} \neq b_{jj}$. Then $a_{jj}=f(\lambda)h(\lambda)$ and $b_{jj}=g(\lambda)h(\lambda)$ where $(f(\lambda), g(\lambda))=1$. Since $m_{jj}=p_{jj}a_{jj}=q_{jj}b_{jj}$ it follows that $g(\lambda)$ divides p_{jj} and $f(\lambda)$ divides q_{jj}. Because $\prod_{m=j}^{n} a_{mm} = \prod_{m=j}^{n} b_{mm}$, it follows that some factor of $g(\lambda)$ divides some element a_{pp} $(p>j)$, and some factor of $f(\lambda)$ divides some element b_{kk} $(k>j)$. That is, $(b_{kk}, q_{jj}) \neq 1$ $(k>j)$ and $(a_{pp}, p_{jj}) \neq 1$ $(p>j)$. By Theorem 1, there exists a family of solutions whose corresponding canonical triangular forms have the same main diagonal as A, and another family, whose corresponding canonical triangular forms have the same main diagonal as B.

If there is a family of solutions, let X_1 be one of the solutions. All members of the family will satisfy the scalar equation $|\lambda I - X_1| = 0$. A scalar equation of finite degree may have only a finite number of dissimilar solutions [2]. Therefore, there must exist at least two similar solutions. Furthermore, there must be at least one class of similar solutions which is infinite in number. This completes the proof of Theorem 3.

3. The solution of $\sum_{m=0}^{r} R_m \cdot \times X^m = 0$. If X_1 is a solution of the unilateral equation and X_2 is a matrix similar to X_1, it does not follow that X_2 is a solution. Since $X_2 = SX_1S^{-1}$, X_2 will be a solution if and only if $\sum_{m=0}^{r} R_m S X^m = 0$. A nonzero, but not necessarily nonsingular, solution for S will exist if and only if $\left| \sum_{m=0}^{r} R_m \cdot \times (X_1^T)^m \right| = 0$ (where $A \cdot \times B = (Ab_{ij})$ and X^T is the transpose of X) [2]. The form of the matrix involved in this determinant creates an interesting problem namely: When does the equation $\sum_{m=0}^{r} R_m \cdot \times X^m = 0$ have a solution? It should be noticed that X need no longer be of order n.

THEOREM 4. *The equation* $\sum_{m=0}^{r} R_m \cdot \times X^m = 0$, *where* R_m $(m=0, 1, \cdots, r)$ *and* X *have elements in* $\mathfrak{F}$, *has a solution if and only if the elements of* $\sum_{m=0}^{r} R_m \lambda^m$ *have a greatest common divisor which is not a unit.*

Consider the matrix $R_m \cdot \times X^m = (R_m x_{ij}^{(m)})$ where $R_m = (r_{ij}^{(m)})$ and $X^m = (x_{ij}^{(m)})$. By simply rearranging its elements this matrix may be changed into the matrix $(r_{ij}^{(m)} X^m)$. If $\sum_{m=0}^{r} R_m \cdot \times X^m = 0$, then $\sum_{m=0}^{r} (r_{ij}^{(m)} X^m) = 0$ and conversely. Therefore, $\sum_{m=0}^{r} r_{ij}^{(m)} X^m = 0$, and the solution X must satisfy the scalar equation $\sum_{m=0}^{r} r_{ij}^{(m)} X^m = 0$, for each value of i and j. Hence the elements of the matrix $\sum_{m=0}^{r} R_m \lambda^m$

must have the minimum polynomial of X as a common divisor.

If the elements of $\sum_{m=0}^{r} R_m\lambda^m$ have a greatest common divisor which is not a unit, then the companion matrix of that polynomial will be a solution of the equations $\sum_{m=0}^{r} r_{ij}^{(m)} X^m = 0$ and hence will be a solution of $\sum_{m=0}^{r} R_m \cdot \times X^m = 0$. Similar considerations show that the same condition is necessary and sufficient for the solution of $\sum_{m=0}^{r} X^m \cdot \times R_m = 0$. A solution of $\sum_{m=0}^{r} R_m \cdot \times X^m = 0$ is a solution of $\sum_{m=0}^{r} X^m \cdot \times R_m = 0$ and conversely.

COROLLARY. *The equations* $\sum_{m=0}^{r} R_m \cdot \times X^m = 0$ *and* $\sum_{m=0}^{r} X^m \cdot \times R_m = 0$ *will have a solution of order* n *if and only if the scalar equation formed by setting the g.c.d. of the elements of* $\sum_{m=0}^{r} R_m\lambda$ *equal to zero has a solution of order* n.

The above condition will certainly be true if the elements of $\sum_{m=0}^{r} R_m\lambda^m$ have a common divisor whose degree is a divisor of n.

THEOREM 5. *If* X_1 *is a solution of the unilateral matrix equation* $\sum_{m=0}^{r} R_m X^m = 0$, *then all matrices similar to* X_1 *are solutions if and only if the minimal polynomial of* X_1 *is a common divisor of the elements of* $\sum_{m=0}^{r} R_m\lambda^m$.

If the minimal polynomial $g(\lambda)$ of X_1 is a common divisor of the elements of $\sum_{m=0}^{r} R_m\lambda^m$, it follows that $\sum_{m=0}^{r} R_m\lambda^m = Q(\lambda)[g(\lambda)I]$.

If X_2 is similar to X_1 then $g(X_2) = 0$ or $g(\lambda)I = P(\lambda)(\lambda I - X_2)$. That is, $\sum_{m=0}^{r} R_m\lambda^m = Q(\lambda)P(\lambda)(\lambda I - X_2)$ and X_2 is a solution of the unilateral equation.

To prove the converse, let $X_2 = SX_1S^{-1}$. Since every matrix similar to X_1 is a solution, it follows that S is a solution of $\sum_{m=0}^{r} R_m S X_1^m = 0$. This is true for every nonsingular S.

The equation $\sum_{m=0}^{r} R_m S X_1^m = 0$ is equivalent to n^2 equations in the unknowns s_{ij} $(i, j = 1, 2, \cdots, n)$. The matrix S may be considered as a vector in n^2 space. If there are n^2 nonsingular solutions S of $\sum_{m=0}^{r} R_m S X_1^m = 0$, and if these matrices, considered as vectors in n^2 space, are linearly independent, then it follows that $\sum_{m=0}^{r} R_m \cdot \times (X_1^T)^m \equiv 0$.

Let E_{ij} be the $n \times n$ matrix which has 1 in the ith row and jth column and zeros elsewhere. The $n^2 - n$ matrices $I + E_{ij}$ $(i \neq j;\ i, j = 1, 2, \cdots, n)$ are linearly independent and nonsingular. The n matrices $\sum_{m \neq i} E_{mm} + a_i E_{ii}$ $(i = 1, 2, \cdots, n)$ are nonsingular and will be linearly independent if the a_i $(i = 1, 2, \cdots, n)$ are chosen so that the determinant of $S_n = \sum_{m=1}^{n} a_m E_{mm} + \sum_{m,p} E_{mp}$ $(m \neq p;\ m, p = 1, 2, \cdots, n)$ is not zero. This is obviously possible when $n = 2$. A simple induction step suffices to show that it is possible for a general n.

A total of n^2 linearly independent nonsingular matrices have been constructed above. If every X_2 similar to X_1 is a solution of $\sum_{m=0}^{r} R_m X^m = 0$, then $\sum_{m=0}^{r} R_m S X_1^m = 0$ has n^2 linearly independent solutions for S. Therefore, $\sum_{m=0}^{r} R_m \cdot \times (X_1^T)^m \equiv 0$, that is, X_1^T is a solution of $\sum_{m=0}^{r} R_m \cdot \times X^m = 0$. Theorem 5 follows from Theorem 4 since the minimal polynomial of X_1^T is the minimal polynomial of X_1.

4. Infinite sets of solutions which are not families. There may be an infinite number of solutions of the unilateral equation without there being an infinite family. This fact is brought out in the proof of the following theorem.

THEOREM 6. *If* $\left| \sum_{m=0}^{r} R_m \lambda^m \right| \equiv 0$, *and if there exists a solution* X_1 *of the unilateral equation* $\sum_{m=0}^{r} R_m X^m = 0$, *then there is an infinite number of solutions.*

The canonical triangular matrix P of $\sum_{m=0}^{r} R_m \lambda^m$ may be factored into the product QA where A is the canonical triangular form of $\lambda I - X_1$, and Q is triangular. Since $\left| \sum_{m=0}^{r} R_m \lambda^m \right| = \delta \prod_{m=1}^{n} p_{mm} = 0$ (δ a unit), it follows that some element $p_{ii} = 0$, and therefore $q_{ii} = 0$.

Factor P into $Q'A'$ letting $a'_{mm} = a_{mm}$ $(m = 1, 2, \cdots, n)$, then $q'_{mm} = q_{mm}$. Let q'_{kk} be the first diagonal element of Q' which is zero. Three possibilities arise in the factorization:

1. $(q'_{jj}, a'_{pp}) \neq 1$, for some values p and j such that $p > j$, and $q_{jj} \neq 0$.
2. Case 1 does not occur and $k < n$.
3. Case 1 does not occur and $k = n$.

In Case 1, as a result of Theorem 1, there exists an infinite family of solutions of the unilateral equation. The fact that some elements q'_{ii} are zero may be taken care of by letting $a'_{l,l+m} = a_{l,l+m}$.

If $q'_{kk} = 0$, that is $p_{kk} = 0$, $(k < n)$, then $p_{km} = 0$ $(m = k, k+1, \cdots, n)$. Since $a'_{mm} \neq 0$ $(m = 1, 2, \cdots, n)$ it follows that all $q'_{km} = 0$. Choose $a'_{lm} = a_{lm}$ $(l, m = 1, 2, \cdots, n-1;\ l = n;\ m = n, n-1, \cdots, k+1)$. The element a'_{kn} is obtained from the equation $\sum_{m=k}^{n} q'_{km} a'_{mn} = 0$. Since all $q'_{km} = 0$, a'_{kn} may be chosen to be a_{kn} with the coefficients of the powers of λ replaced by parameters. Since $(q'_{mm}, a'_{nn}) = 1$ $(m < k)$, the remaining terms of A' may be expressed uniquely in terms of a'_{kn}. As in Theorem 1, it follows that there is a family of right divisors of P with the same diagonal as A, and hence a family of solutions of the unilateral equation. In the final case $k = n$, and $(q'_{mm}, a_{pp}) = 1 (m < p)$. Choose $a'_{lm} = a_{lm}$ $(l, m = 1, 2, \cdots, n-1)$. If a'_{nn} is chosen to be a_{nn}, a unique factorization is obtained namely $P = QA$. Since $q'_{nn} = 0$, a'_{nn} may be chosen arbitrarily. Choose a'_{nn} to be identical with a_{nn} except

in the constant term which will be a parameter.

It may easily be shown that if there is given a finite set of irreducible polynomials, and a given polynomial of fixed degree with an arbitrary constant, then for an infinite set of values of the arbitrary constant, the polynomial will be relatively prime to all of the polynomials in the given finite set. This follows from an application of Euclid's algorithm and the fact that the field is of characteristic zero.

If the finite set of irreducible polynomials is taken to be the factors of the elements q'_{mm} $(m=1, 2, \cdots, n-1)$, then for an infinite set of values of the constant $(q'_{mm}, a'_{nn})=1$. Since $(q'_{mm}, a_{nn})=1$, a_{nn} will be one of the a'_{nn}. A unique factorization of P exists for each a'_{nn}. Therefore, there is an infinite set of right divisors of P. Since A is included in the set, there is an infinite set of right divisors of P which are left associates of monic matrices of degree 1. It follows that there is an infinite set of solutions of the unilateral equation.

Note that in Case 3 the set of solutions, as defined, is not a family.

It may be possible in certain cases to factor P in such a way as to get a family of right divisors which leads to a family of solutions. That this is not always true is shown by the following example in which $\mathfrak{F}$ is the rational field R. Let

$$P = \begin{Vmatrix} \lambda^2 & 1 \\ 0 & 0 \end{Vmatrix}.$$

There is no factorization of P in $R[\lambda]$ which will yield an infinite family of right divisors, and solutions of the equation:

$$\begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix} X^2 + \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix} = \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix}.$$

However, since

$$\begin{Vmatrix} \lambda^2 & 1 \\ 0 & 0 \end{Vmatrix} = \begin{Vmatrix} \lambda & 1/e \\ 0 & 0 \end{Vmatrix} \begin{Vmatrix} \lambda & -1/e \\ 0 & \lambda + e \end{Vmatrix}, \qquad e \neq 0,$$

there is the infinite set of solutions:

$$X = \begin{Vmatrix} 0 & 1/e \\ 0 & -e \end{Vmatrix}.$$

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CANONICAL POSITIVE DEFINITE MATRICES UNDER INTERNAL LINEAR TRANSFORMATIONS

BERNARD VINOGRADE

Let P be a real positive definite n-rowed square matrix, $n=p+q+r$, $p\leqq q\leqq r$, $S=\text{diag}\ (S_1, S_2, S_3)$, where S_1 is p-rowed, S_2 is q-rowed, and S_3 is r-rowed, all nonsingular. S is called an internal linear transformation when it transforms P according to the rule $P\rightarrow SPS'$. All coefficients are real numbers. The problem is to find a canonical form for P, under internal linear transformations, depending on characteristic value systems associated with P. The partition of n into three parts is for convenience only.

1. **Semi-canonical reduction.** Partition P according to the partition of n, and call the square diagonal blocks P_1, P_2, and P_3, with p, q, and r rows respectively. Choose $S_1=Q_1^{-1}$, $S_2=Q_2^{-1}$, $S_3=Q_3^{-1}$ from the factorizations $P_1=Q_1Q_1'$, $P_2=Q_2Q_2'$, $P_3=Q_3Q_3'$. Then

$$SPS'=Q=\begin{pmatrix} I_p & A & B \\ A' & I_q & C \\ B' & C' & I_r \end{pmatrix}, \tag{1}$$

where I_i is the i-rowed identity matrix. Any further transformations will be chosen so as to preserve the diagonal blocks I_p, I_q, I_r. Hence we shall have $S_iS_i'=I_i$, $i=1, 2, 3$.

2. **Non-special cases.** Let $X=(I_p, A, B)$, $Y=(A', I_q, C)$, $Z=(B', C', I_r)$, so XX', YY', and ZZ' are all nonsingular and symmetric. Let their characteristic roots be the ordered diagonal elements of $D_1=\text{diag}\ (\lambda_1, \cdots, \lambda_p)$, $D_2=\text{diag}\ (\mu_1, \cdots, \mu_q)$, and D_3

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$=\text{diag}\ (\nu_1, \cdots, \nu_r)$. If the λ_i's are distinct, then an orthogonal matrix K_1 such that $K_1XX'K_1' = D_1$ is unique within a left factor $\text{diag}\ (\pm 1, \cdots, \pm 1)$. If also the μ_i's and the ν_i's are distinct respectively, then $S = \text{diag}\ (K_1, K_2, K_3)$ gives SQS' in a canonical form with rows and corresponding columns orthogonal in groups of p, q, and r, and coefficients unique within internal permutations and sign changes.

3. **Special cases.** After carrying out the reduction of §2, let $\text{diag}\ (K_1, K_2, K_3)$ stand for any further orthogonal internal transformation. Suppose there are a distinct λ_i's and $a < p$. For convenience of discussion, assume there are a_1 λ_1's in the first a_1 places in D_1, $\lambda_1 \neq 1$ and $a_1 > 1$. Then in K_1 there is a free orthogonal a_1-rowed block. Generally, $K_1 = \text{diag}\ (K_{11}, \cdots, K_{1a})$, each K_{1i} orthogonal. It follows from $(A, B)(A, B)' = \text{diag}\ (\lambda_1 - 1, \cdots, \lambda_p - 1)$ that if (A_{a_1}, B_{a_1}) are the first a_1 rows of (A, B) then $(A_{a_1}, B_{a_1})(A_{a_1}, B_{a_1})' = (\lambda_1 - 1)I_{a_1}$. Impose on K_{11} the condition $K_{11}A_{a_1}A_{a_1}K_{11}' = \text{diag}\ (h_1, \cdots, h_{a_1})$. If the h_i's are not distinct, suppose $h_1 = \cdots = h_{\alpha_1} \neq 0$. Let L_1 be the corresponding $\alpha_1 \times \alpha_1$ free orthogonal block in K_{11}, and $(A_{\alpha_1}, B_{\alpha_1})$ the corresponding α_1 rows, then A_{α_1} is of rank α_1 and contains an $\alpha_1 \times \alpha_1$ nonsingular minor array G. After an internal permutation suppose $A_{\alpha_1} = (G, H)$, then the free L_1 may be absorbed by requiring L_1G to be the unique triangular factor (say with positive diagonal coefficients) of $G'G$. If $h_1 = 0$ then the process starts with B_{a_1}, and if the partition is into more than three parts, B_{a_1} may be orthogonalized first. If $\lambda_1 = 1$, then clearly $(A_{a_1}, B_{a_1}) = 0$. None of these transformations affect the orthogonality conditions imposed by the reduction of §2. Y and Z are reduced similarly with the freedom in K_2 and K_3, if any.

4. **The case $n = p + q$.** In this case a more detailed description is possible and is derived here as a consequence of the following lemma. It has also been displayed by Hotelling as a by-product of a statistical investigation.[1]

LEMMA. *If two real matrices A and B are each a-rowed and b-columned, then $AA' = BB'$ if and only if $A = BU$ where U is orthogonal.*

PROOF. AA' is an a-rowed symmetric matrix, hence there exists an orthogonal V such that $VAA'V' = VBB'V' = \text{diag}\ (d_1, \cdots, d_r, 0, \cdots, 0)$, where $r \leqq a$ and each d_i is positive. Let $D^{-1} = \text{diag}\ (1/(d_1)^{1/2} \cdots, 1/(d_r)^{1/2}, 1 \cdots, 1)$. Then $D^{-1}VAA'V'D^{-1} = D^{-1}VBB'V'D^{-1}$, $= \text{diag}\ (I_r, 0_{a-r})$. Write $D^{-1}VA = X$ and $D^{-1}VB = Y$, so $XX' = YY'$

[1] H. Hotelling, *Relations between two sets of variates*, Biometrika vol. 28 (1936) p. 321. The present paper is the basis for an extension to more than two sets.

$=\text{diag}\,(I_r, 0_{a-r})$. This implies that X has r nonzero orthonormal rows. Now to the r nonzero rows add $b-r$ rows chosen so as to present a b-rowed orthogonal matrix X_0. Do the same with Y and call the resulting orthogonal matrix Y_0. Letting $U=Y_0'X_0$, we have $X_0=Y_0U$. Hence $X=YU$ and $A=BU$.

Let

$$\begin{pmatrix} I_p & A \\ A' & I_q \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$$

be the semi-canonical form (1) of a positive definite matrix partitioned according to $n=p+q$. Let K be an orthogonal matrix such that $KXX'K'=D$, where D is diagonal. Hence $KAA'K'=D-I_p=BB'$, where B is a p by q matrix whose first p diagonal coefficients are the square roots of those of $D-I_p$. By the lemma there exists an orthogonal matrix L' such that $B=KAL'$. Then the canonical form desired is obtained by using the internal transformation

$$\begin{pmatrix} K & 0 \\ 0 & L \end{pmatrix}.$$

5. **Applications.** These canonical forms have been applied to the determination of minimal principal minors of partitioned correlation matrices in an unpublished paper by R. G. D. Steel and the author. These minors have statistical significance as generalizations of the minors $1-b_i^2$ formed from the matrix of §4, with b_i a diagonal coefficient of B (see footnote 1).

One easily proved property of the semi-canonical reduction of §1 is the following: Among all SPS' with unit diagonal coefficients and fixed[2] P, the determinant $|SPS'|$ is maximum when SPS' is in the semi-canonical form. For, let P_1 be the first diagonal block of P and let SPS' be in semi-canonical form. Then $S_1P_1S_1'=I_p$, or $|S_1|^2=1/|P_1|\geqq 1$. Similarly $|S_2|^2\geqq 1$, $|S_3|^2\geqq 1$. Hence $|S|^2\geqq 1$, implying $|SPS'|\geqq P$.

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[2] Here P is a correlation matrix.

NOTE ON THE ESCALATOR METHOD

BERNARD VINOGRADE

Practical solution of the characteristic vector problem

$$AX = \lambda X, \tag{1}$$

where $A = (a_{ij})$ is a given real nonsingular n-rowed matrix and X is a column $(x_1, \cdots, x_n)'$, has recently been treated anew by J. Morris.[1] His process is called the escalator method. If A_p is a p-rowed principal minor array in the upper left corner of A, then the abbreviated system

$$A_pX_p = \lambda X_p, \qquad p \leqq n, \tag{2}$$

is called the pth stage of (1). As given by Morris, the escalator method is based on expressions for the solutions of the $(p+1)$th stage in terms of those of the pth stage, and depends on the existence of biorthonormal sets of solution vectors for (2) and the transposed system

$$A_p'Y_p = \lambda Y_p. \tag{3}$$

But, of course, it is not always possible to find such biorthonormal sets of solution vectors, since the elementary divisors of the pth stage may not be linear. In this note a more general exposition of escalation is given, covering all cases and leading to at least one practical method in cases outside the scope of Morris' formulas. In addition, a result on the transmission of roots from one stage to the next is included.

Let λ_i be a characteristic root of multiplicity n_i of the system (2), and X_{pi} a corresponding characteristic vector. As is well known, there exists a nonsingular P_0 such that

$$A_pP_0 = P_0A_0, \quad \text{where} \quad A_0 = \begin{pmatrix} \lambda_1 & \delta_1 & & & & 0 \\ & \lambda_1 & \delta_2 & & & \\ & & \ddots & \ddots & & \\ & & & \lambda_1 & 0 & \\ & & & & \lambda_2 & \delta_{n_1} \\ & & & & & \ddots \\ 0 & & & & & \end{pmatrix}, \tag{4}$$

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[1] J. Morris, *The escalator method*, Wiley, 1947.

the δ_i's being zero or one. If we set

$$C = (a_{p+1,1}, \cdots, a_{p+1,p}), \quad D = (a_{1,p+1}, \cdots, a_{p,p+1})',$$

$$(5) \qquad A_{p+1} = \begin{pmatrix} A_p & D \\ C & a_{p+1,p+1} \end{pmatrix}, \quad P = \begin{pmatrix} P_0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = P^{-1}A_{p+1}P,$$

then one can easily expand the equation

$$(6) \quad |A_{p+1} - \lambda I_{p+1}| = |B - \lambda I_{p+1}| = \begin{vmatrix} A_0 - \lambda I_p & P_0^{-1}D \\ CP_0 & a_{p+1,p+1} - \lambda \end{vmatrix} = 0,$$

where I_p is the p-rowed identity matrix. This equation is the general escalator equation for the determination of the characteristic roots at the $(p+1)$th stage. When A_0 is diagonal, then (6) coincides with Morris' equation[2] and is especially effective in symmetric systems. Since $A_p' P_0^{-1\prime} = P_0^{-1\prime} A_0'$, the transposed system (3) leads to the same escalator equation $|B' - \lambda I_{p+1}| = 0$.

For a determination of P_0, one may place X_{p1} in the first column, X_{p2} in the (n_2+1)th column, and so on. The remaining coefficients and the δ_i's may then be solved from the system (4).[3]

For a determination of the characteristic vectors of the $(p+1)$th stage,[4] let X^* be a $p+1$-columned matrix of the unknown vectors and let Λ be the diagonal matrix of corresponding characteristic roots of the $(p+1)$th stage as determined from the escalator equation (6). Then from $A_{p+1}X^* = X^*\Lambda$ it follows that $P^{-1}A_{p+1}PP^{-1}X^* = P^{-1}X^*\Lambda$, or $BZ^* = Z^*\Lambda$, where $Z^* = P^{-1}X^*$ and $B = P^{-1}A_{p+1}P$ (as defined by (5)). Because of the simple structure of B, the equations $BZ^* = Z^*\Lambda$ afford an easy determination of Z^*, hence of $X^* = PZ^*$.

As for the transposed system $A_{p+1}'Y^* = Y^*\Lambda$, we note that if Z^* is nonsingular (which is possible whenever the elementary divisors at the $(p+1)$th stage are linear), it follows from $BZ^* = Z^*\Lambda$ that $B'Z^{*-1\prime} = Z^{*-1\prime}\Lambda$, hence $Y^* = P^{-1\prime}Z^{*-1\prime} = X^{*-1\prime}$ is a solution of the transposed system.

If $CP_0 = (P_1, \cdots, P_p)$, $P_0^{-1}D = (Q_1, \cdots, Q_p)'$, then a necessary and sufficient condition for a root λ_r of the pth stage to be a root of the $(p+1)$th stage is $P_rQ_r = 0$ for any pair P_r, Q_r arising from a column of P_0 which is a characteristic vector of the pth stage cor-

[2] The components of CP_0 and $P_0^{-1}D$ are Morris' P_i's and P_i's.

[3] In connection with (4), the escalator method for simultaneous linear systems as compactly presented by R. A. Frazer, *Note on the escalator method*, Philosophical Magazine vol. 38 (1947) pp. 287–289, is pertinent.

[4] Checks and other special procedures have been developed by R. J. Lambert, Master's Thesis, Iowa State College of Agriculture and Mechanic Arts, 1948.

responding to λ_r.

For the necessity, let x_{p+1} be the last component of X_{p+1} and notice that the systems $A_{p+1}X_{p+1}=\lambda X_{p+1}$ and $A'_{p+1}Y_{p+1}=\lambda Y_{p+1}$ may be written

$$(A_p - \lambda I_p)X_p + Dx_{p+1} = 0, \tag{7}$$
$$CX_p + x_{p+1}(a_{p+1,p+1} - \lambda) = 0,$$

$$(A'_p - \lambda I_p)Y_p + C'y_{p+1} = 0, \tag{8}$$
$$D'Y_p + y_{p+1}(a_{p+1,p+1} - \lambda) = 0.$$

Hence if $P_r=CX_{pr}=0$, then (7) is satisfied by $\lambda=\lambda_r$ and $X_{p+1}=(X'_{pr}, 0)'$. Similarly, $Q_r=0$ implies (8) is satisfied by $\lambda=\lambda_r$ and $Y_{p+1}=(Y'_{pr}, 0)'$.

On the other hand, if we left-multiply the p equations $(A_p, D)X_{p+1}=\lambda X_p$ by Y'_{pr} and recall that $A'_p Y_{pr}=\lambda_r Y_{pr}$, then

$$(\lambda_r - \lambda)Y'_{pr}X_p + Q_r x_{p+1} = 0. \tag{9}$$

Now if (9) is satisfied by $\lambda=\lambda_r$ and a corresponding vector $X_{p+1,r}$, then either $Q_r=0$ or $x_{p+1}=0$. If $x_{p+1}=0$, then the first p components of $X_{p+1,r}$ satisfy the pth stage, hence $P_r=0$. Thus the sufficiency is proved.

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THE MAXIMAL REGULAR IDEAL OF A RING

BAILEY BROWN AND NEAL H. McCOY

1. Introduction. An element a of a ring R is said to be *regular* if and only if there exists an element x of R such that $axa=a$. The ring R is *regular* if and only if each element of R is regular. The concept of a regular ring was introduced by von Neumann [5, 6][1] who, however, required also that a regular ring have a unit element.

Unless otherwise stated, the word *ideal* shall mean two-sided ideal, and an ideal in R will be said to be regular if and only if it consists entirely of regular elements of R. It is easy to see that a regular ideal A in R is itself a regular ring. For if $a \in A$, there exists an element x of R such that $axa=a$. It follows that $axaxa=a$ and $xax \in A$, so a is regular in the ring A.

We shall show that the join of all regular ideals in an arbitrary ring R is a regular ideal, and hence that there exists a unique maximal regular ideal $M=M(R)$ in R. The purpose of this note is to establish a few fundamental properties of $M(R)$. Among these are the following "radical-like" properties: (i) $M(R/M(R))=0$, (ii) if B is an ideal in R, then $M(B)=B \cap M(R)$, (iii) if R_n is the complete matrix ring of order n over R, then $M(R_n)=(M(R))_n$. A special case of this last result is that R_n is regular if and only if R is regular. This was proved by von Neumann [6], but we shall include a very simple proof of this fact.

It is well known that every regular ring has zero (Jacobson) radical J. For the rest of the introduction it is assumed that R is a ring such that R/J is regular. We note that this condition is satisfied if, for example, the right ideals of R satisfy the descending chain condition. Thus $M=R$ if and only if $J=0$, and hence, in some sense, M may be considered as an "anti-radical." It is shown in §4 that $M=0$ if and only if R is bound to its radical J in the sense of Marshall Hall [2]. Moreover, in §5 it is proved that, under the descending chain condition for right ideals, R is expressible as a direct sum

$$R = M + M^*,$$

where M^* is the ideal consisting of all elements a of R such that $aM=Ma=0$. It follows that M is semi-simple and M^* is bound to its radical, and thus this direct sum decomposition coincides with one

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

obtained by Marshall Hall [2].

2. **Existence and simple properties of $M(R)$.** Let R be an arbitrary ring, and a an element of R. The following lemma plays a central role in several of our proofs:

LEMMA 1. *If y is an element of R such that $a-aya$ is regular, then a is regular.*

PROOF. If $a-aya$ is regular, there exists an element z of R such that

$$(a - aya)z(a - aya) = a - aya.$$

If we set $x=z-zay-yaz+yazay+y$, a simple calculation shows that $axa=a$, and thus a is regular, which completes the proof.

We shall indicate by (a) the principal ideal in R generated by a. We now prove the following theorem.

THEOREM 1. *If M is the set of all elements a of R such that (a) is regular, then M is an ideal in R.*

PROOF. If $z\in M$ and $t\in R$, then $zt\in M$ since $(zt)\subseteq(z)$. Similarly, $tz\in M$. If z, $w\in M$ and $a\in(z-w)$, then $a=u-v$ for some u in (z) and v in (w). Since (z) is regular, $u=uru$ for some element r of R. Then

$$a - ara = u - v - (u - v)r(u - v) = -v + urv + vru - vrv.$$

Since $v\in(w)$, this shows that $a-ara\in(w)$ and is therefore regular. Lemma 1 now implies that a is regular, and hence $z-w\in M$. This completes the proof of the theorem.

It is clear that M, being the join of all regular ideals in R, and being itself regular, is the unique maximal regular ideal in R. It may be remarked that the proof of the above theorem is analogous to the proof of Theorem 1 in Brown and McCoy [1].

We shall next prove the following theorem.

THEOREM 2. *If R is any ring, $M(R/M(R))=0$.*

PROOF. Let $\bar{a}$ denote the residue class modulo $M(R)$ which contains the element a of R. If $\bar{b}\in M(R/M(R))$ and $a\in(b)$, then $\bar{a}\in(\bar{b})$. Since $(\bar{b})$ is a regular ideal in $R/M(R)$, $\bar{a}$ is regular. If $\bar{a}=\bar{a}\bar{x}\bar{a}$, $a-axa\in M(R)$, therefore $a-axa$ is regular and Lemma 1 implies that a is regular. This shows that every element of (b) is regular, and hence $b\in M(R)$. Thus $\bar{b}=0$, completing the proof.

Suppose now that B is an ideal in R, and let b be an element of B which generates a regular ideal $(b)'$ in the ring B. Let (b) be the ideal in R generated by the element b, and let

$$c = nb + rb + bs + \sum r_i b s_i \qquad (n \text{ an integer; } r, s, r_i, s_i \text{ in } R)$$

be any element of (b). Since b is regular in B we have $b = bb_1b$ for some b_1 in B. Hence

$$c = nb + (rbb_1)b + b(b_1bs) + \sum (r_i b b_1) b (b_1 b s_i),$$

and thus $c \in (b)'$; therefore (b) is regular since it coincides with $(b)'$. This shows that if $b \in M(B)$, then $b \in B \cap M(R)$. Conversely, if $b \in B \cap M(R)$, then b is an element of B which is regular in R, and it is easy to see that b is therefore regular in the ring B. Since $B \cap M(R)$ is a regular ideal in the ring B, it follows that $B \cap M(R) \subseteq M(B)$. We have therefore proved the following theorem.

THEOREM 3. *If B is an ideal in R, then $M(B) = B \cap M(R)$.*

3. The maximal regular ideal of a complete matrix ring. In this section we shall prove the following theorem.

THEOREM 4. *If R_n is the complete matrix ring of order n over R, then*

$$M(R_n) = (M(R))_n.$$

First we give an elementary proof of the special case of this result in which R itself is regular, and therefore $R = M(R)$. This result, under the assumption that R has a unit element, is due to von Neumann [6].

LEMMA 2. *If R is a regular ring, then R_n is a regular ring.*

The proof of this is in two steps, the first being the proof for $n = 2$, and the second the extension to arbitrary n. If $r \in R$, let us denote by r' an element of R such that $rr'r = r$. Now let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be an arbitrary element of R_2. If we set

$$X = \begin{pmatrix} 0 & 0 \\ b' & 0 \end{pmatrix},$$

and denote $A - AXA$ by B, a simple calculation shows that

$$B = \begin{pmatrix} g & 0 \\ h & i \end{pmatrix}$$

for suitable choice of elements g, h, i of R. If

$$Y = \begin{pmatrix} t' & 0 \\ 0 & i' \end{pmatrix},$$

then

$$C = B - BYB = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix},$$

for some element k of R. Finally, if

$$Z = \begin{pmatrix} 0 & k' \\ 0 & 0 \end{pmatrix},$$

we see that

$$C - CZC = 0.$$

This means that C is regular and hence, by Lemma 1, B is regular. Again applying Lemma 1, we see that A is regular, and this completes the proof for $n=2$.

Since $(R_2)_2 \cong R_4$, it follows from the case just proved that R_4 is regular, and similarly R_{2^k} is regular for any positive integer k. If now n is an arbitrary positive integer, choose k so that $2^k \geqq n$. If $A \in R_n$, let A_1 be the matrix of R_{2^k} with A in the upper left-hand corner and zeros elsewhere. Now, as an element of R_{2^k}, A_1 is regular, that is, there exists an element

$$X = \begin{pmatrix} B & C \\ D & E \end{pmatrix}, \qquad (B \in R_n),$$

of R_{2^k} such that $A_1XA_1 = A_1$. However, this implies that $ABA = A$, and hence A is regular. The proof of the lemma is therefore complete.

By the lemma just proved, $(M(R))_n$ is a regular ideal in R_n, and hence $(M(R))_n \subseteq M(R_n)$. Conversely, let A be a matrix in $M(R_n)$, and let a_{ij} be a fixed element of A. Since (A) is a regular ideal, there exists an element X of R_n such that $A = AXA = AXAXA$, and therefore

$$a_{ij} = \sum_{p,q} t_{pq} a_{pq} s_{pq}$$

for suitable elements t_{pq}, s_{pq} of R. But it is easy to see[2] that there exists a matrix of (A) with $t_{pq}a_{pq}s_{pq}$ in (1, 1) position and zeros elsewhere, and hence an element of (A) with a_{ij} in (1,1) position and zeros elsewhere. Now if b is any element of the principal ideal in R

[2] See, for example, Lemma 5 of [1].

generated by a_{ij}, it is clear that there exists an element B of (A) with b in the (1,1) position and zeros elsewhere. Furthermore, we have $BYB=B$ for suitable choice of Y in R_n since (A) is regular. But this implies that $by_{11}b=b$, and hence b is regular. This shows that $a_{ij} \in M(R)$, and hence that $M(R_n) \subseteq (M(R))_n$, completing the proof of the theorem.

4. Some additional properties of $M(R)$. By the *annihilator* B^* of an ideal B in a ring R is meant the ideal consisting of all elements a of R such that $aB=Ba=0$.

THEOREM 5. *If M is the maximal regular ideal of a ring R and J is the Jacobson radical of R, then $M \cap J=0$, $J \subseteq M^*$, $M \subseteq J^*$, and $M \cap M^*=0$. Furthermore, J is the radical of the ring M^* and M is the maximal regular ideal of the ring J^*.*

PROOF. Since J contains no nonzero idempotent element [3, p. 305], $M \cap J=0$. From this it follows that $MJ=JM=0$, so $J \subseteq M^*$ and $M \subseteq J^*$. If $a \in M \cap M^*$, then $a=axa$ for some x. But $a \in M$ and $xa \in M^*$, hence $a(xa)=0$ and $M \cap M^*=0$. The last sentence of the theorem follows from the observation of Perlis [4] that if B is any ideal in R, the radical of the ring B is just $B \cap J$, and from the analogous Theorem 3.

Following Marshall Hall [2], we may say that a ring R is *bound* to its radical J if and only if $J^* \subseteq J$.

The next theorem gives, for a class of rings including all those whose right ideals satisfy the descending chain condition, a necessary and sufficient condition that the maximal regular ideal be the zero ideal.

THEOREM 6. *If R is a ring such that R/J is regular, then $M=0$ if and only if R is bound to J.*

PROOF. If R is bound to J, it follows that $M=0$ even without the condition that R/J be regular. For $M \cap J=0$, and this implies, as in Theorem 5, that $M \subseteq J^* \subseteq J$. Hence $M=0$.

Conversely, let R/J be regular and $M=0$. We show first by induction that $J \cap J^{*2}=0$. Suppose that $j \in J$ and that $j=\sum_{i=1}^{n} a_i b_i$ where a_i, b_i are in J^*. It must be proved that $j=0$. In the regular ring R/J, $\bar{a}_i$ is regular, so R contains x_i such that $a_i - a_i x_i a_i = j_i \in J$. Since $b_i \in J^*$, we have

$$j = \sum_{i=1}^{n} (a_i x_i a_i + j_i) b_i = \sum_{i=1}^{n} a_i x_i a_i b_i. \tag{1}$$

If $n=1$, this implies that $j=a_1x_1a_1b_1=a_1x_1j=0$ since $a_1 \in J^*$. If $n \neq 1$,

then

$$a_n b_n = j - \sum_{i=1}^{n-1} a_i b_i.$$

Thus by (1)

$$j = \sum_{i=1}^{n-1} a_i x_i a_i b_i + a_n x_n \left(j - \sum_{i=1}^{n-1} a_i b_i \right) = \sum_{i=1}^{n-1} (a_i x_i - a_n x_n) a_i b_i.$$

But the induction hypothesis asserts that if $j = \sum_{i=1}^{n-1} c_i d_i$ and c_i, d_i are in J^*, then $j=0$. Since $(a_i x_i - a_n x_n)a_i$ and b_i are in J^*, it follows that $j=0$, and we have proved that $J \cap J^{*2} = 0$. This implies, however, that J^{*2} is a regular ideal. For if $a \in J^{*2}$, then in the regular ring R/J, the element $\bar{a}$ is regular, that is, for some x, $a - axa \in J \cap J^{*2} = 0$, so a is regular. Hence $J^{*2} \subseteq M = 0$, from which it follows that $J^* \subseteq J$ since the radical contains all nil ideals [3, p. 304]. Thus R is bound to J and the proof is complete.

5. **A decomposition theorem.** In this section we point out the role played by the maximal regular ideal M in a theorem of Hall [2], and incidentally give a new proof of his result.

LEMMA 3. *If an ideal B in a ring R has a unit element e, then*

$$R = B + B^*.$$

PROOF. The existence of a unit element in B implies that $B \cap B^* = 0$. If $x \in R$, then $ex + xe \in B$ and hence $(ex+xe)e = e(ex+xe)$, from which it follows that $xe = ex$ and e is in the center of R. Thus the Peirce decomposition

$$x = ex + (x - ex)$$

expresses each element x of R as a sum of elements ex of B and $x - ex$ of B^*, and the desired result is established.

We remark that a right ideal I in the ring M is a right ideal in R. For if $a \in I$, $r \in R$, then $ar \in M$, hence for some element y of R, $aryar = ar$. But $ryar \in M$, so $ar \in I$. Thus I is a right ideal in R.

From this remark, it follows that if the descending chain condition for right ideals holds in R, it holds also in M. In the presence of this chain condition, regularity is equivalent to semi-simplicity. Hence M has a unit element, and the first sentence of the following theorem is implied directly by Lemma 3.

THEOREM 7. *If a ring R satisfies the descending chain condition for right ideals, then*

$$R = M + M^*.$$

The ring M is semi-simple and the ring M^* is bound to its radical.

The semi-simplicity of M is implied by the regularity of M. Since the maximal regular ideal of M^* is zero by Theorem 3, and the chain condition holds in M^*, it follows from Theorem 6 that M^* is bound to its radical.

Hall has shown that a ring R satisfying the descending chain condition for right ideals can be represented in a unique way as the direct sum of a semi-simple ring and a ring which is bound to its radical. The result just established shows that the semi-simple component is precisely the maximal regular ideal M of R, and the bound component is the annihilator of M.

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PROOF OF RAMANUJAN'S PARTITION CONGRUENCE FOR THE MODULUS 11^3

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1. Introduction. Ramanujan's conjecture, that

$$p(11^\alpha l + \rho) \equiv 0 \pmod{11^\alpha}, \qquad l = 0, 1, 2, \cdots; \alpha = 1, 2, \cdots \tag{1.1}$$

where $p(n)$ is the number of partitions of n and ρ is the least positive integer[2] such that

$$24\rho \equiv 1 \pmod{11^\alpha}, \tag{1.2}$$

has never been proved for general values of α. Proofs for $\alpha=1, 2$ were given by Ramanujan and the author.[3] The object of this note is to supply a proof for $\alpha=3$, that is, to establish the congruence

$$p(1331l + 721) \equiv 0 \pmod{11^3}, \qquad l = 0, 1, 2, \cdots. \tag{1.3}$$

The proof in I for $\alpha=1$ used a method developed by Rademacher (Trans. Amer. Math. Soc. vol. 51 (1942) pp. 609–636). It was effected by constructing an identity which, by analogy, would be of the form

$$\sum_{l=0}^{\infty} p(11^\alpha l + \rho) x^{l+1} = 11^\alpha \sum \tag{1.4}$$

for general α. Here $\sum$ should be a sum, with integer coefficients, of certain functions having expansions in powers of x with integral coefficients. Now the power series on the left-hand side of (1.4) can be expressed rather simply in terms of the Dedekind function

$$\begin{aligned} \eta(\tau) &= \exp(\pi i\tau/12) \prod_1^\infty (1 - x^m) \\ &= \exp(\pi i\tau/12) \left\{1 + \sum_1^\infty p(n) x^n\right\}^{-1}, \end{aligned} \tag{1.5}$$

where, throughout this paper, $x = \exp 2\pi i\tau$ and $\operatorname{Im} \tau > 0$. In fact, it is

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[1] The author is greatly indebted to the referee, who made a very careful review of the paper, correcting some errors and clarifying a number of ambiguities. He also supplied footnote 2.

[2] Actually ρ can be given explicitly. It is $(23 \cdot 11^\alpha + 1)/24$ for α even and $(13 \cdot 11^\alpha + 1)/24$ for α odd.

[3] Cf. footnotes 2 and 5 in J. Lehner, *Ramanujan identities involving the partition function for the moduli* 11^α, Amer. J. Math. (1943) pp. 492–520. This paper will be referred to hereafter as I.

easy to deduce (I, (7.32)) from the above definition that

$$\prod_{m=1}^{\infty}(1-x^{\epsilon m})\sum_{l=0}^{\infty}p(11^{\alpha}l+\rho)x^{l+1} = 11^{-\alpha}\sum_{\lambda \bmod 11^{\alpha}}\eta(\epsilon\tau)\eta^{-1}((\tau+24\lambda)/11^{\alpha}) = L(\tau;11^{\alpha}), \tag{1.6}$$

say, where ϵ is 1 for even α and 11 for odd α.

The important fact about $L(\tau;11^{\alpha})$ is that (I, Theorem 8) it is a modular function on the subgroup[4] $\Gamma_0(11)$ of the full modular group, and has its only singularity at the parabolic vertex $\tau=0$ of the subgroup's fundamental region. As such it must be a polynomial of the form

$$Q_{\alpha}(A,C) = (\gamma_0+\gamma_1C+\cdots)+AC(\delta_0+\delta_1C+\cdots) \tag{1.7}$$

in the two basic functions $A(\tau)$ and $C(\tau)$, whose definitions we shall recall in §3. Thus we have

$$\prod_{m=1}^{\infty}(1-x^{\epsilon m})\sum_{l=0}^{\infty}p(11^{\alpha}l+\rho)x^{l+1} = L(\tau;11^{\alpha}) = Q_{\alpha}(A,C). \tag{1.8}$$

Now the power series of $A(\tau)$ and $C(\tau)$ in terms of x have integral coefficients, and so Ramanujan's conjecture for 11^{α} follows if we can show that the coefficients of the polynomial Q_{α} are divisible by 11^{α}. This was done in I for $\alpha=1$ by explicit calculation.

To go on to higher values of α we proved that the functions $L(\tau;11^{\alpha})$ can be given inductively by using the linear operator

$$U_{11}F(\tau) = UF(\tau) = 11^{-1}\sum_{\lambda=0}^{10}F((\tau+\lambda)/11). \tag{1.9}$$

In fact (I, Theorem 7), if we put $L(\tau;11^0)=1$, we have

$$L(\tau;11^{2m+1}) = U\{\Phi(\tau)\cdot L(\tau;11^{2m})\}, \tag{1.10}$$

$$L(\tau;11^{2m+2}) = UL(\tau;11^{2m+1}), \tag{1.11}$$

for $m=0, 1, 2, \cdots$, where

$$\Phi(\tau) = \Phi(\tau;121) = \eta(121\tau)\eta^{-1}(\tau). \tag{1.12}$$

The nature of the inductive step depends on the parity of α.

Now UC^j, $U\Phi C^{j-1}$, UAC^j, $U\Phi AC^j$, $j=1, 2, \cdots$, are modular functions on $\Gamma_0(11)$ whose only singularities are at the parabolic vertex zero (I, Theorem 8). Thus, because U is linear, we can go from

[4] Defined by the condition $11|c$ in the modular substitution $\tau'=(a\tau+b)/(c\tau+d)$.

one of the identities (1.8) to the next provided we know the expressions for $UC^i, \cdots$, as polynomials in A and C. However, if we are interested only in proving Ramanujan's conjecture for a given α, we can abbreviate this explicit procedure considerably, as was done in I for $\alpha=2$ and will be done below for $\alpha=3$. The basic tool will still be the inductive property (1.10), (1.11).

2. **Method.** We shall find it convenient to replace the rational basis $A(\tau)$, $C(\tau)$ by the basis $C(\tau)$, $D(\tau)$, where

$$11D(\tau) = A(\tau)C(\tau) - 1, \tag{2.1}$$

and $D(\tau)$ has integral coefficients (I, Lemma 4). A modular function on $\Gamma_0(11)$ whose only singularity is at the parabolic vertex $\tau=0$ (for example, $L(\tau; 11^\alpha)$) can be expressed as a polynomial in C and D which is of first degree in D.

Now let us use the notation $F(\tau)\equiv G(\tau) \pmod{11^\alpha}$ to mean that corresponding coefficients of the expansions of $F(\tau)$ and $G(\tau)$ about $i\infty$ in powers of x are congruent mod 11^α. Then proving Ramanujan's conjecture for $\alpha=3$ is equivalent to proving that

$$L(\tau; 11^3) \equiv 0 \pmod{11^3}. \tag{2.2}$$

The operator U affects a power series as follows (I, (4.74)):

$$U\left(\sum_{n\geqq\nu} a_n x^n\right) = \sum_{m\geqq\nu/11} a_{11m}x^m. \tag{2.3}$$

Thus we can say that if the coefficients of the expansion of $F(\tau)$ are integral, then so are the coefficients in the expansion of $UF(\tau)$. Hence from $F(\tau)\equiv G(\tau) \pmod{11^\alpha}$, we can infer that $UF(\tau)\equiv UG(\tau) \pmod{11^\alpha}$.

The identity for $L(\tau; 11)$ is explicitly (I, (6.6))

$$L(\tau; 11) = 11AC(11C + 2) - 11(11^2C^2 + 32C + 2),$$

or in terms of C, D

$$L(\tau; 11) = 11^2D(11C + 2) - 11(11^2C^2 + 21C). \tag{2.4}$$

Hence

$$L(\tau; 11) \equiv 2\cdot 11^2D - 21\cdot 11C \pmod{11^3}. \tag{2.5}$$

Now we apply the second inductive step (1.11) and make use of the remark in the preceding paragraph and the linearity of U to obtain

$$L(\tau; 11^2) = UL(\tau; 11) \equiv 2\cdot 11^2UD - 21\cdot 11UC \pmod{11^3}. \tag{2.6}$$

In the next section we shall prove

(A) $$UC \equiv 11a_1 C \pmod{11^2}, \qquad a_1 = \text{integer},$$

(B) $$UD \equiv 0 \pmod{11}.$$

From these congruences, the truth of which we assume for the moment, we conclude from (2.6) that

(2.7) $$L(\tau; 11^2) \equiv 11^2 a_1 C \pmod{11^3}.$$

Now $\Phi(\tau)$ has an expansion in powers of x with integral coefficients, as we easily verify from (1.12) and (1.5). Hence, if we apply the first inductive step (1.10) to (2.7), we find

(2.8) $$L(\tau; 11^3) \equiv 11^2 a_1 U\Phi C \pmod{11^3}.$$

Therefore, the desired congruence (1.3) will be completely proved if we can establish (A), (B), and

(C) $$U\Phi C \equiv 0 \pmod{11}.$$

3. **Proof of (A), (B), and (C).** In the proof of these equations, we shall make use of the fact, demonstrated in I, §8, that their left members, namely, UC, UD, $U\Phi C$, are entire[5] modular functions on the subgroup $\Gamma_0(11)$ of the modular group, and shall express them as polynomials in the basic functions C, D of this subgroup. We therefore recall the definition of C and D. As before, τ is a complex variable with positive imaginary part, and $x = \exp 2\pi i\tau$. All the functions listed below have Laurent expansions in x with integral coefficients. For further details, cf. I, §4.

Starting with the formula for $\eta(\tau)$ in (1.5), we define in turn:

(3.1) $$\psi(\tau) = \{\eta(\tau)\eta(11\tau)\}^3,$$

(3.2) $$g(\tau) - 1/120 = 2\sum_1^\infty \sigma_3(n)x^n, \qquad \sigma_3(n) = \sum_{d|n} d^3,$$

(3.3) $$G(\tau) = 11^3 g(11\tau) - g(\tau),$$

(3.4) $$\theta(\tau) = \sum_{m,n=-\infty}^{+\infty} x^{Q(m,n)}, \qquad Q = m^2 + mn + 3n^2,$$

(3.5) $$A(\tau) = \theta^3(\tau)/\psi(\tau),$$

(3.6) $$B(\tau) = G(\tau)/\psi^2(\tau),$$

(3.7) $$C(\tau) = 2^{-1}\cdot 11^{-2}\{A^3(\tau) - 10A(\tau) - 22 - B(\tau)\}.$$

[5] An *entire* modular function is one which has no singularities except possibly at the parabolic vertices of the fundamental region, which, in this particular case, are located at 0 and $i\infty$.

The expansions of the following functions of interest at the parabolic vertices $(0, i\infty)$ of $\Gamma_0(11)$ appear in I (references at right of display lines are to equations in I). In order to avoid confusion we write $(UF|\tau')$ to denote the result of replacing τ by τ' in the right member of (1.9). By $P(x^s)$ we shall mean a series in powers of x, with integral coefficients, beginning with a term in x^s.

$$A(\tau) = x^{-1} + 6 + \cdots \tag{3.8}$$ (6.41),

$$C(\tau) = x + 5x^2 + \cdots \tag{3.9}$$ (6.43),

$$A(-1/11\tau) = A(\tau) = x^{-1} + 6 + \cdots \tag{3.10}$$ (4.53),

$$11^2C(-1/11\tau) = x^{-2} + 2x^{-1} + \cdots \tag{3.11}$$ (6.46),

$$UC(\tau) = P(x), \tag{3.12}$$ (8.2),

$$11^3(UC\,|\,-1/11\tau) = 11^2C(-1/121\tau) + 11^2P(x) = x^{-22} + \cdots \tag{3.13}$$ (8.81), (9.31).

The first terms of the expansions of D can be deduced from (2.1), (3.8)–(3.11):

$$D(\tau) = x + \cdots, \tag{3.14}$$

$$11^3D(-1/11\tau) = x^{-3} + \cdots. \tag{3.15}$$

By Fermat's Theorem and an elementary theorem on binomial coefficients, we observe that

$$11^2C(-1/121\tau) = \{11^2C(-1/11\tau)\}^{11} + 11P(x^{-21}); \tag{3.16}$$

consequently, in view of (3.13),

$$11^3(UC\,|\,-1/11\tau) = \{11^2C(-1/11\tau)\}^{11} + 11P(x^{-21}). \tag{3.17}$$

The argument in the first paragraph on p. 505 of I shows that an entire modular function on $\Gamma_0(11)$ which vanishes at infinity and has at most a pole of first order at the vertex $\tau=0$ must be identically zero. We apply this principle to the function

$$\Delta(\tau) = 11^3UC(\tau) - \{11^2C(\tau)\}^{11} - \Big\{11^3\alpha_1C(\tau) + 11^4\beta_0D(\tau) + \sum_{j=2}^{10} 11^{2j+1}\alpha_jC^j(\tau) + 11^3D(\tau)\cdot\sum_{j=1}^{9} 11^{2j+1}\beta_jC^j(\tau)\Big\},$$

where the α, β are rational integers chosen so that all terms of the principal part of Δ at $\tau' = -1/11\tau = 0$ drop out except possibly the term in x^{-1}. This choice is clearly possible in view of (3.17), (3.11), and (3.15). Furthermore, all terms of Δ vanish at $\tau = i\infty$, as we see from (3.9), (3.12), (3.14), and therefore Δ itself. It follows that $\Delta(\tau)$

is identically zero, and we infer the truth of the congruence

$$UC(\tau) \equiv \alpha_1 C(\tau) + 11\beta_0 D(\tau) \pmod{11^2}. \tag{3.18}$$

Now writing

$$C(\tau) = \sum_1^\infty c_n x^n, \qquad D(\tau) = \sum_1^\infty d_n x^n, \qquad C^2(\tau) = \sum_2^\infty c_n^{(2)} x^n, \tag{3.19}$$

and remembering (2.3), we obtain the following linear congruences for α_1 and β_0 on matching the coefficients of x and x^2 in the two members of (3.18):

$$\begin{aligned} c_1\alpha_1 + 11d_1\beta_0 &\equiv c_{11} \\ c_2\alpha_1 + 11d_2\beta_0 &\equiv c_{22} \end{aligned} \pmod{11^2},$$

or using the values in the tables of §4,

$$\begin{aligned} \alpha_1 + 11\beta_0 &\equiv 44 \\ 5\alpha_1 + 66\beta_0 &\equiv 99 \end{aligned} \pmod{11^2}. \tag{3.20}$$

Therefore, we have

$$\begin{aligned} \alpha_1 &\equiv 6(44) - 99 \pmod{11^2}, \\ \alpha_1 &\equiv 0 \pmod{11}; \end{aligned} \tag{3.21}$$

$$\begin{aligned} 11\beta_0 &\equiv 99 - 5(44) = 11(9 - 20) \pmod{11^2}, \\ \beta_0 &\equiv 0 \pmod{11}. \end{aligned} \tag{3.22}$$

With (3.18), (3.21), and (3.22), we have completed the proof of (A).

The proof of (B) is quite similar. From I, (8.81), and (3.14), (3.15),

$$\begin{aligned} 11^4(UD\,|\,-1/11\tau) &= 11^3 D(-1/121\tau) + 11^3 P(x) \\ &= x^{-33} + \cdots, \end{aligned} \tag{3.23}$$

and, as before,

$$11^4(UD\,|\,-1/11\tau) = \{11^3 D(-1/11\tau)\}^{11} + 11P(x^{-32}). \tag{3.24}$$

Now employing the same procedure used for UC, we obtain easily

$$11^4 UD(\tau) \equiv \alpha_1 11^3 C(\tau) + \beta_0 11^4 D(\tau) \pmod{11^5},$$

with integral α_1, β_0. Equating coefficients of x, we have $\alpha_1 + 11\beta_0 \equiv 0 \pmod{11}$, that is, $\alpha_1 = 11\alpha$. Hence,

$$UD \equiv \alpha C + \beta_0 D \pmod{11}, \qquad \alpha, \beta_0 \text{ integral.} \tag{3.25}$$

We now again equate coefficients and get the congruences

$$\begin{aligned} \alpha + \beta_0 &\equiv d_{11} \\ 5\alpha + 6\beta_0 &\equiv d_{22} \end{aligned} \pmod{11} \tag{3.26}$$

for the determination of α and β_0. The determinant of the coefficients is seen to be equal to unity. Hence

$$\alpha \equiv \begin{vmatrix} d_{11} & d_1 \\ d_{22} & d_2 \end{vmatrix} \equiv 0 \pmod{11}, \tag{3.27}$$

since $d_{11} \equiv d_{22} \equiv 0 \pmod{11}$. Likewise,

$$\beta_0 \equiv \begin{vmatrix} c_1 & d_{11} \\ c_2 & d_{22} \end{vmatrix} \equiv 0 \pmod{11}. \tag{3.28}$$

This proves (B).

For the proof of (C), we need the expansions

$$\Phi(\tau) = x^5 + \cdots \tag{3.29}$$

$$11\Phi(-1/121\tau) = \Phi^{-1}(\tau) = x^{-5} + \cdots . \tag{3.30}$$

The first is computed immediately from (1.12) and (1.5), whereas (3.30) is a consequence of the well known transformation equation

$$\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau), \tag{3.31}$$

where the square root is positive when τ is on the positive imaginary axis. Then by virtue of (2.3) we may write

$$U\Phi C = P(x). \tag{3.32}$$

We must now find the expansion of $U\Phi C$ at $\tau' = 0$, that is, $(U\Phi C \mid -1/11\tau)$. We cannot use I, (8.81) this time, for ΦC is not a function of $\Gamma_0(11)$. Instead, we have recourse to I, p. 509, in which the function $G(\tau; i, j, k) = U\Phi^i A^j C^k$ is considered. (The last two display lines on p. 509 are in error: the line at the bottom of the page should have the factor i multiplying the right member; the display line above this requires the factor[6] $1/\Phi(\tau/11)$ affecting the sum in the right member.) The case we are interested in is $j = 0$, $i = k = 1$. If we substitute the value of the sum given at the bottom of the page in the expression for $11G(-1/\tau)$ appearing immediately above, and replace τ by 11τ, we obtain

$$\begin{aligned} 11^4(U\Phi C \mid -1/11\tau) = &\left\{11\Phi(-1/121\tau)\right\}\left\{11^2 C(-1/121\tau)\right\} \\ &- 11^3\Phi^{-1}(\tau) \sum_{m=5}^{\infty} \sum_{p=1}^{\infty} \phi_m c_p \left(\frac{24(m+p)}{11}\right) \exp\left\{2\pi i(m+p)\tau\right\}, \end{aligned}$$

[6] This factor is also required multiplying the sum in the right member of (8.31).

with the Legendre-Jacobi symbol

$$\left(\frac{24(m+p)}{11}\right).$$

From the last equation we deduce, in view of (3.29), (3.30), and (3.11),

$$11^4(U\Phi C\,|\,-1/11\tau) = \{11\Phi(-1/121\tau)\}\{11^2C(-1/121\tau)\} + 11^3P(x) = x^{-27} + \cdots. \tag{3.33}$$

We therefore obtain in the usual way[7] a congruence

$$11^4U\Phi C = \alpha_1 11^2C + \alpha_2 11C^2 + 11^3\beta_0 D \pmod{11^5} \tag{3.34}$$

with rational integers α, β. We want to show that $11^2 \mid \alpha_1$, $11 \mid \beta_0$. This, however, is a consequence of the following congruences obtained by matching the coefficients of x and x^2 in the two members of (3.34):

$$\begin{aligned} \alpha_1 + 11\beta_0 &\equiv 0 \\ 5\alpha_1 + 66\beta_0 &\equiv 0 \end{aligned} \pmod{11^2}. \tag{3.35}$$

Therefore, $\alpha_1 = 11^2\alpha$, $\beta_0 = 11\beta$, and we have from (3.34)

$$U\Phi C \equiv \alpha C + \alpha_2 C^2 + \beta D \pmod{11}, \qquad \alpha, \alpha_2, \beta \text{ integral.} \tag{3.36}$$

(3.36), in turn, yields the congruences

$$\begin{aligned} \alpha \quad\quad + \quad \beta &\equiv e_{11} \\ 5\alpha + \quad \alpha_2 + \quad 6\beta &\equiv e_{22} \\ 19\alpha + 10\alpha_2 + 28\beta &\equiv e_{33} \end{aligned} \pmod{11}, \tag{3.37}$$

where we have written

$$\Phi(\tau)C(\tau) = \sum_{n=0}^{\infty} e_n x^n. \tag{3.38}$$

The determinant of the coefficients is -1. Since $e_{11} \equiv e_{22} \equiv e_{33} \equiv 0 \pmod{11}$, we have

$$\alpha \equiv -\begin{vmatrix} e_{11} & 0 & 1 \\ e_{22} & 1 & 6 \\ e_{33} & 10 & 28 \end{vmatrix} \equiv 0 \pmod{11},$$

[7] The analogue of (3.16) and (3.24) is not valid for the function ΦC, since $\Phi(\tau)$ is not a function of $\Gamma_0(11)$. The equation we would like to write, $11\Phi(-1/121\tau) \cdot 11^2C(-1/121\tau) = \{11\Phi(-1/11\tau) \cdot 11^2C(-1/11\tau)\}^{11} + 11P(x^{-33})$, is obviously false, for (3.30) shows that $\Phi(-1/11\tau)$ does not have an expansion in integral powers of x.

and likewise $\alpha_3 \equiv \beta \equiv 0 \pmod{11}$. This establishes (C), and with it, the truth of the congruence (1.3).

4. Tables. The preceeding discussion has utilized the values of certain coefficients, which will be developed in this section. In the computation of these, the following formulae, in addition to those given previously, will be found useful:

$$\prod_1^{\infty}(1-x^m) = \sum_{-\infty}^{+\infty}(-1)^n x^{n(3n+1)/2} \qquad \text{(Euler)}, \tag{4.1}$$

$$\prod_1^{\infty}(1-x^m)^{-1} = 1 + \sum_1^{\infty} p(n)x^n \qquad \text{(Euler)}, \tag{4.2}$$

$$\theta(\tau) = \sum_0^{\infty} A_Q(n)x^n \qquad \text{(I, (4.61) and footnote 16)} \tag{4.3}$$

with[8]

$$A_Q(n) = \begin{cases} 2\sum_{d_1|n} 1 - 2\sum_{d_2|n} 1, & 11 \nmid n, \\ A_Q(n/11), & 11 \mid n, \end{cases}$$

where $d_1 \equiv$ quad. res., $d_2 \equiv$ quad. non-res. (mod 11).

The coefficients of the functions C, D, and ΦC are computed in a straightforward way from the definitions (1.5), (1.12), (3.1)–(3.7), (2.1), with the help of (4.1)–(4.3). The values are shown in the following tables to the highest modulus required for the proofs of the preceding section.

RESIDUE OF THE COEFFICIENTS OF CERTAIN POWER SERIES

Power of x	Function									
	ψ^{-1}	θ^3	A	A^3	ψ^{-3}	G	B	C	D	Φ
	modulo 0									
−2				1	1		1			
−1	1		1	12	4		2			
0	2	1	6	70	14	1	− 12			
1	5	4	17	296	40	− 2	− 116	1	1	
2	10	4	46	1,073	105	−18	− 597	5	6	
3	20	8	116	3,460	252	−56	−2,298	19	28	

[8] The formula for $A_Q(n)$ was printed incorrectly in I.

Power of x	Function									
	ψ^{-1}	θ^2	A	A^3	ψ^{-3}	G	B	C	D	Φ
	modulo $2\cdot 11^4$							mod 11^3	mod 11	
4	36	20	252	10,150	574	− 146	− 7,616	63		
5	65	16	533	27,704	1,240	− 252	−22,396	64		1
6	110	32	1,034	12,528	2,580	− 504	27,114	18		1
7	185	16	1,961	26,858	5,180	− 688	22,010	60		2
8	300	36	3,540	23,282	10,108	− 1,170	12,082	21		3
9	481	28	6,253	27,176	19,212	− 1,514	6,028	71		5
10	754	40	10,654	207	6,427	− 2,268	18,539	89		7
11	1,169	4	17,897	23,232	6,452	− 2,422	9,306	44		0
12	1,780	64	2	25,108	28,404	− 4,088	4,276	86		4
13	2,685	40	17,983	3,456	28,712	− 4,396	770	115		0
14	3,996	64	16,304	20,971	2,641	− 6,192	24,911	36		8
15	5,894	56	30	13,030	19,004	− 7,056	16,602	105		9
16	8,600	68	4,916	1,420	23,952	− 9,362	18,326	90		1
17	12,450	40	12,024	20,348	364	− 9,828	12,154	21		0
18	17,860	100	5,352	609	17,447	−13,626	23,561	47		2
19	25,442	48	5,288	15,722	26,482	−13,720	7,612	57		3
20	6,682	104	9,118	29,184	23,816	10,886	3,102	94		0
21	21,237	80	26,561	28,386	1,102	10,018	3,324	95		0
22	11,926	4	21,362	5,929	17,083	7,484	2,607	99		0
23	9,954	56	8,592	5,160	22,928	4,946	22,574	57		0
	modulo $2\cdot 11^3$							modulo 11		
24	1,792	144	46	2,376	220	1,846	1,190	3		6
25	1,421	68	1,435	1,718	2,174	442	1,888	6		0
26	2,436	88	310	893	1,447	366	2,391	3		0
27	2,544	104	1,118	2,562	138	1,712	94	8		1
28	1,012	128	858	384	1,040	354	758	7		1
29	2,125	72	745			1,798				2
30		176				384				0

$d_{11}=d_{21}=0 \pmod{11}$, $c_1^{(3)}=0$, $c_2^{(3)}=1$, $c_3^{(3)}=10$
$e_{11}=e_{12}=e_{31}=0 \pmod{11}$, where $\Phi C=\sum e_n x^n$
$\phi_{21}=5$, $\phi_{22}=7 \pmod{11}$

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A GENERAL ASYMPTOTIC FORMULA FOR PARTITION FUNCTIONS

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1. **Introduction.** Practically all partition functions that have been considered have generating functions of the form

$$g_\gamma(s) = \prod_{n=1}^{\infty} \frac{1}{(1-e^{-sn})^{\gamma(n)}} = \sum_{k=0}^{\infty} a_\gamma(k) e^{-sk}, \tag{1}$$

where the real part of s is positive and

$$F_\gamma(x) = \sum_{n \leqq x} \gamma(n) \sim K x^u \log^v x, \qquad \gamma(n) \geqq 0,\ K > 0,\ u > 0. \tag{2}$$

Here $a_\gamma(0) = 1$ and for k positive

$$a_\gamma(k) = \sum_{k=1k_1+2k_2+\cdots+lk_l,\, k_n \neq 0} \prod_{n=1}^{l} \frac{\gamma(n)[\gamma(n)+1]\cdots[\gamma(n)+k_n-1]}{k_n!}. \tag{3}$$

Thus the partition $k = 1k_1 + 2k_2 + \cdots + lk_l$ of k into k_1 ones, k_2 twos, and so on, is weighted in a certain way by the product shown, and various partition functions are obtained by choosing the function $\gamma(n)$ appropriately. For example, if $\gamma(n) = 1$ for all n, then $a_\gamma(k)$ becomes the unrestricted partition function; if $\gamma(n) = n$ for all n, then $a_\gamma(k)$ becomes the plane partition of Wright [2];[1] if $\gamma(n) = 1$ for rth powers and 0 otherwise, then $a_\gamma(k)$ is the number of partitions of k into rth powers; and so on.

The purpose of this paper is to establish the asymptotic formula

$$\log A_\gamma(k) \sim \frac{1}{u} \{Ku\Gamma(u+2)\zeta(u+1)\}^{1/(u+1)} \cdot \{u+1\}^{(u-v)/(u+1)} k^{u/(u+1)} \{\log k\}^{v/(u+1)}, \tag{4}$$

where

$$A_\gamma(k) = \sum_{n=0}^{k} a_\gamma(n). \tag{5}$$

We prove (4) by a method used by Hardy and Ramanujan [1] to obtain some special cases thereof, namely, we first prove that for s

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[1] Numbers in brackets refer to the references at the end of the paper.

tending to zero through positive values

$$\log g_\gamma(s) \sim K\left(\frac{1}{s}\right)^u\left(\log\frac{1}{s}\right)^v\Gamma(u+1)\zeta(u+1), \tag{6}$$

and then apply a Tauberian theorem proved by Hardy and Ramanujan in their paper.

If $a_\gamma(k)$ is nondecreasing with increasing k from a certain point on,

$$\frac{A_\gamma(k)+O(1)}{k} \leqq a_\gamma(k) \leqq A_\gamma(k) \tag{7}$$

and hence (4) is true for $\log a_\gamma(k)$. This is the case, for example, if $\gamma(1)\geqq 1$. Formula (4) is obviously true for $a_\gamma(k)$ under wider conditions than monotonicity, but on the other hand it is not always true for $a_\gamma(k)$; for example, it is obviously false if $\gamma(n)$ is 1 for even n and 0 for odd n.

Furthermore:[2] Suppose that $\gamma(n)\geqq 1$ whenever it is nonzero. Suppose that the set S of positive integers for which $\gamma(n)\neq 0$ has the property that all positive integers not less than c can be partitioned into integers from S. Then (4) is also true for $\log a_\gamma(k)$.

To prove this observe that $a_\gamma(k)\geqq a_\gamma(m)$ for $k-c\geqq m\geqq 1$. Hence

$$A_\gamma(k-c)/(k-c) \leqq a_\gamma(k) \leqq A_\gamma(k) \qquad \text{for } k > c$$

and the conclusion follows from (4).

As a result of this, one can deduce asymptotic formulas for the logarithm of any unweighted partition function, provided only that the set of integers used is such that all sufficiently large integers can be partitioned into integers from the set. In particular, this is certainly true if the set of positive integers contains two coprime integers.

An important lacuna in the literature on partitions is the absence of an exact or asymptotic formula for the number $T(k)$ of partitions of k into primes. From (4) and the prime number theorem we get only

$$\log T(k) \sim \frac{2\pi}{3^{1/2}}\left(\frac{k}{\log k}\right)^{1/2} \tag{8}$$

which is given by Hardy and Ramanujan (formula 5.281 of [1]).

In the paper following this one we consider a weighted partition function $a(k)$ involving primes, namely that obtained by choosing $\gamma(n)$ in (1) as the $\Lambda(n)$ of prime number theory. We obtain there an

[2] The author is indebted to the referee for this observation, which materially extends the scope of the paper.

asymptotic formula for $A(k)=\sum_{n=0}^{k} a(n)$ under the assumption of the Riemann hypothesis. From (4) follows the weaker result

$$\log A(k) \sim \pi(2k/3)^{1/2} \tag{9}$$

without any unproved hypothesis. Since proof of monotonicity of $a(k)$ would seem to be extremely difficult (if true), we cannot assert (9) for $\log a(k)$. Note that (9) is the same as the asymptotic formula for $\log p(k)$, where $p(k)$ is the number of unrestricted partitions of k. The correspondence follows from the fact that

$$\sum_{n\leq x} \Lambda(n) \sim x \sim \sum_{n\leq x} 1.$$

Since also

$$\sum_{p\leq x} \log p \sim x,$$

the powers of the primes may be disregarded in (9).

Besides (8) and (9) we give some further applications of (4) at the end of the paper.

2. **Proof.** In the following s is real and positive. Both "o" and "O" refer to s tending to zero.

From (1) we obtain

$$\log g_\gamma(s) = \sum_{n=1}^{\infty} \gamma(n) \log \frac{1}{1-e^{-sn}} = \sum_{m=1}^{\infty} \frac{1}{m} \phi(ms)$$

where

$$\phi(s) = \sum_{n=1}^{\infty} \gamma(n) e^{-sn}.$$

We find an asymptotic formula for $\phi(s)$ for s tending to zero, and use it to find an asymptotic formula for $\log g_\gamma(s)$. This is the previously mentioned technique of Hardy and Ramanujan [1, pp. 245–261].

$$\begin{aligned}
\phi(s) &= \gamma(1)e^{-s} + \gamma(2)e^{-2s} + \gamma(3)e^{-3s} + \gamma(4)e^{-4s} + \cdots \\
&= -\gamma(1)\{e^{-2s} - e^{-s}\} - \{\gamma(1)+\gamma(2)\}\{e^{-3s} - e^{-2s}\} \\
&\quad - \{\gamma(1)+\gamma(2)+\gamma(3)\}\{e^{-4s} - e^{-3s}\} - \cdots \\
&= -\sum_{q=1}^{\infty} \sum_{n=1}^{q} \gamma(n) \int_{q}^{q+1} \frac{d(e^{-sy})}{dy}\, dy
\end{aligned}$$

$$= \sum_{q=1}^{\infty} \int_{q}^{q+1} F_\gamma(y) s e^{-sy} dy$$

$$= \int_{1}^{\infty} F_\gamma(y) s e^{-sy} dy = \int_{s}^{\infty} F_\gamma\left(\frac{x}{s}\right) e^{-x} dx.$$

Thus

$$\phi(s) = \int_{s}^{s^{1/2}} F_\gamma\left(\frac{x}{s}\right) e^{-x} dx + \int_{s^{1/2}}^{\infty} F_\gamma\left(\frac{x}{s}\right) e^{-x} dx. \tag{10}$$

$F_\gamma(x)$ is an increasing function, hence in the first integral of (10),

$$F_\gamma\left(\frac{x}{s}\right) \leqq F_\gamma\left(\frac{1}{s^{1/2}}\right) \leqq C s^{-u/2} \log^v \frac{1}{s}, \quad \text{for } s < \frac{1}{e},$$

and so

$$\int_{s}^{s^{1/2}} F_\gamma\left(\frac{x}{s}\right) e^{-x} dx = O\left\{ s^{(1-u)/2} \log^v \frac{1}{s} \right\}, \quad \text{for } s \text{ tending to zero.}$$

s may be taken so small that, in the second integral of (10),

$$F_\gamma\left(\frac{x}{s}\right) = K\left(\frac{x}{s}\right)^u \left(\log \frac{x}{s}\right)^v (1 + \rho_1(x))$$

where $|\rho_1(x)| < \epsilon$ uniformly in x, $x \geqq s^{1/2}$. Then:

$$\phi(s) = O\left\{ s^{(1-u)/2} \log^v \frac{1}{s} \right\} + (1 + o(1)) K \int_{s^{1/2}}^{\infty} \left(\frac{x}{s}\right)^u \left(\log \frac{x}{s}\right)^v e^{-x} dx. \tag{11}$$

Put

$$\Phi(s) = \int_{s^{1/2}}^{\infty} \left(\frac{x}{s}\right)^u \left(\log \frac{x}{s}\right)^v e^{-x} dx = \int_{s^{1/2}}^{1/s^{1/2}} + \int_{1/s^{1/2}}^{\infty} = \Phi_1(s) + \Phi_2(s).$$

For $\Phi_1(s)$, $-1/2 \leqq \log x / \log (1/s) \leqq 1/2$ and

$$\log^v \frac{x}{s} = \left(\log^v \frac{1}{s}\right)\left(1 + \frac{\log x}{\log (1/s)}\right)^v$$

$$= \left(\log^v \frac{1}{s}\right)\left(1 + \frac{\log x}{\log (1/s)} \rho_2(x)\right),$$

where $|\rho_2(x)| \leqq C_v$. Therefore,

$$\Phi_1(s) = \left(\frac{1}{s}\right)^u \left(\log\frac{1}{s}\right)^v \int_{s^{1/2}}^{1/s^{1/2}} x^u \left(1 + \frac{\log x}{\log(1/s)}\rho_2(x)\right) e^{-x}dx$$

$$= \left(\frac{1}{s}\right)^u \left(\log\frac{1}{s}\right)^v \int_0^\infty x^u e^{-x}dx - \left(\frac{1}{s}\right)^u \left(\log\frac{1}{s}\right)^v \int_0^{s^{1/2}} x^u e^{-x}dx$$

$$- \left(\frac{1}{s}\right)^u \left(\log\frac{1}{s}\right)^v \int_{1/s^{1/2}}^\infty x^u e^{-x}dx$$

$$+ \left(\frac{1}{s}\right)^u \left(\log\frac{1}{s}\right)^{v-1} \int_{s^{1/2}}^{1/s^{1/2}} x^u (\log x)\rho_2(x)e^{-x}dx.$$

We find:

$$\Phi_1(s) = \left(\frac{1}{s}\right)^u \left(\log\frac{1}{s}\right)^v \Gamma(u+1) + O\left\{s^{(1-u)/2}\log^v\frac{1}{s}\right\}$$

$$+ o(1) + O\left\{\left(\frac{1}{s}\right)^u \left(\log\frac{1}{s}\right)^{v-1}\right\}.$$

Also, $\Phi_2(s) = o(1)$.

Substituting $\Phi(s) = \Phi_1(s) + \Phi_2(s)$ into (11), we have

$$\phi(s) = O\left\{s^{(1-u)/2}\log^v\frac{1}{s}\right\}$$

$$+ \{1 + o(1)\}\left\{K\left(\frac{1}{s}\right)^u \left(\log\frac{1}{s}\right)^v \Gamma(u+1)\right\}$$

$$+ O\left\{s^{(1-u)/2}\log^v\frac{1}{s}\right\} + o(1)$$

$$+ O\left\{\left(\frac{1}{s}\right)^u \left(\log\frac{1}{s}\right)^{v-1}\right\} + o(1).$$

Hence

$$\phi(s) \sim K\left(\frac{1}{s}\right)^u \left(\log\frac{1}{s}\right)^v \Gamma(u+1).$$

Now

$$\log g_\gamma(s) = \sum_{m=1}^\infty \frac{1}{m}\phi(ms) = \sum_{1\le m\le M} + \sum_{M<m<1/s^{1/2}} + \sum_{1/s^{1/2}\le m\le 1/s} + \sum_{m>1/s}$$

$$= \phi_1(s) + \phi_2(s) + \phi_3(s) + \phi_4(s),$$

where M is such that $\sum_{m>M} 1/m^{u+1} < \epsilon$ ($M(\epsilon)$ is fixed).

First:

$$\phi_1(s) = \sum_{m=1}^{M} \frac{1}{m} \phi(ms)$$

$$= \sum_{m=1}^{M} \frac{1}{m} \{1 + o(1)\} \left\{ K \left(\frac{1}{ms}\right)^u \left(\log \frac{1}{ms}\right)^v \Gamma(u+1) \right\}.$$

But $\log m / \log (1/s) < 1/2$ for $\phi_1(s)$ and

$$\left(\log \frac{1}{ms}\right)^v = \left(\log \frac{1}{s}\right)^v \left(1 - \frac{\log m}{\log (1/s)}\right)^v$$

$$= \left(\log \frac{1}{s}\right)^v \left(1 + \frac{\log m}{\log (1/s)} R_v(m)\right)$$

where $| R_v(m) | < C_v$.

Thus

$$\phi_1(s) = K \left(\frac{1}{s}\right)^u \left(\log \frac{1}{s}\right)^v \Gamma(u+1) \sum_{m=1}^{M} \frac{1 + o(1)}{m^{u+1}}$$

$$+ K \left(\frac{1}{s}\right)^u \left(\log \frac{1}{s}\right)^{v-1} \Gamma(u+1) \sum_{m=1}^{M} \frac{R_v(m) \log m}{m^{u+1}} \{1 + o(1)\}.$$

But M is fixed, and so

$$\phi_1(s) = K \left(\frac{1}{s}\right)^u \left(\log \frac{1}{s}\right)^v \Gamma(u+1) \zeta(u+1) \{1 + o(1)\}$$

$$+ O\left\{ \epsilon \left(\frac{1}{s}\right)^u \left(\log \frac{1}{s}\right)^v \right\} + O\left\{ \left(\frac{1}{s}\right)^u \left(\log \frac{1}{s}\right)^{v-1} \right\}.$$

Secondly, for $\phi_2(s)$ we have

$$\phi(ms) < C_{uv} \left(\frac{1}{ms}\right)^u \left(\log \frac{1}{ms}\right)^v,$$

and since $M < m < 1/s^{1/2}$

$$\left(\log \frac{1}{ms}\right)^v \leqq C_v \left(\log \frac{1}{s}\right)^v \qquad \text{for all } v.$$

So,

$$\phi_2(s) = O\left\{ \epsilon \left(\frac{1}{s}\right)^u \left(\log \frac{1}{s}\right)^v \right\}.$$

Thirdly, for $\phi_3(s)$ we observe from the original definition that $\phi(m_1 s) < \phi(m_2 s)$ if $m_1 > m_2$. Hence we may use for all m the estimation for the smallest m, namely $m = 1/s^{1/2}$, and find

$$\phi_3(s) < C_{u,v}\left(\frac{1}{s}\right)^{u/2}\left(\log\frac{1}{s}\right)^{v}\sum_{1}^{1/s}\frac{1}{m}$$

or,

$$\phi_3(s) = O\left\{\left(\frac{1}{s}\right)^{u/2}\left(\log\frac{1}{s}\right)^{v+1}\right\}.$$

Finally, in $\phi_4(s)$, $ms > 1$, and the definition of $\phi(ms)$ readily yields $\phi_4(s) = O(1)$. See (3.544) of Hardy-Ramanujan.

This gives

$$\begin{aligned}\log g_7(s) = {} & K\left(\frac{1}{s}\right)^{u}\left(\log\frac{1}{s}\right)^{v}\Gamma(u+1)\zeta(u+1)\{1+o(1)\}\\ & + O\left\{\epsilon\left(\frac{1}{s}\right)^{u}\left(\log\frac{1}{s}\right)^{v}\right\}\\ & + O\left\{\left(\frac{1}{s}\right)^{u}\left(\log\frac{1}{s}\right)^{v-1}\right\}\\ & + O\left\{\epsilon\left(\frac{1}{s}\right)^{u}\left(\log\frac{1}{s}\right)^{v}\right\}\\ & + O\left\{\left(\frac{1}{s}\right)^{u/2}\left(\log\frac{1}{s}\right)^{v+1}\right\}\\ & + O\{1\},\end{aligned}$$

and we have

$$\log g_7(s) \sim K\left(\frac{1}{s}\right)^{u}\left(\log\frac{1}{s}\right)^{v}\Gamma(u+1)\zeta(u+1). \tag{6}$$

Now we quote, for reference, Hardy and Ramanujan's Theorem A [1, p. 252]:

"Suppose that

$$\lambda_1 \geqq 0,\ \lambda_n > \lambda_{n-1},\ \lambda_n \to \infty;\ \frac{\lambda_n}{\lambda_{n-1}} \to 1;\ a_n \geqq 0,\ A > 0,\ \alpha > 0;$$

$\sum a_n e^{-\lambda_n s}$ is convergent for $s > 0$;

$$f(s) = \sum a_n e^{-\lambda_n s} = \exp\left[\{1 + o(1)\} A s^{-\alpha} \left\{\log \frac{1}{s}\right\}^{-\beta}\right]$$

when $s \to 0$. Then

$$A_n = a_1 + \cdots + a_n = \exp\left[\{1 + o(1)\} B \lambda_n^{\alpha/(1+\alpha)} (\log \lambda_n)^{-\beta/(1+\alpha)}\right]$$

where $B = A^{1/(1+\alpha)} \alpha^{-\alpha/(1+\alpha)} (1+\alpha)^{1+[\beta/(1+\alpha)]}$ when $n \to \infty$."

Applying this theorem with:

$$f(s) = g_\gamma(s), \qquad A = K\Gamma(u+1)\zeta(u+1),$$
$$\lambda_n = n = k, \qquad \beta = -v,$$
$$\alpha = u,$$

we obtain (4) at once from (6).

3. **Further applications.** A few applications were indicated in the introduction.

As a matter of interest we give some further applications.

A fairly general type of partition is obtained by using only components of the form:

$$n = a_0 + a_1 m + a_2 m^2 + \cdots + a_r m^r, \qquad a_r > 0,\ r \geqq 1, \tag{12}$$

where the a's are such as to give positive integral n for positive integral m. For simplicity we also assume the polynomial to yield $n_1 > n_2$ for $m_1 > m_2$. The m's themselves may be restricted in various ways.

Let m_1 be the largest integer which gives $n \leqq x$ in (12). Then

$$a_0 + \cdots + a_{r-1} m_1^{r-1} + a_r m_1^r \leqq x < a_0 + \cdots + a_{r-1}(m_1+1)^{r-1} + a_r (m_1+1)^r.$$

As $x \to \infty$, $m_1 \to \infty$, and so for x tending to infinity we have

$$x = a_r m_1^r + O(m_1^{r-1}) = a_r m_1^r \left(1 + O\left(\frac{1}{m_1}\right)\right) = a_r m_1^r (1 + o(1)),$$

$$m_1 = \left(\frac{x}{a_r}\right)^{1/r} (1 + o(1)).$$

Let m be restricted in some manner such that

$$\sum_{m \leqq m_1} 1 \sim \kappa m_1^\mu (\log m_1)^\nu, \qquad \mu > 0. \tag{13}$$

Obviously, $\mu \leqq 1$. But when only the restricted m's of (13) are used in (12), $\sum_{m \leqq m_1} 1 = \sum_{n \leqq x} 1$. Thus:

$$\sum_{n\leq x} 1 = \frac{\kappa}{r^\nu}\left(\frac{x}{a_r}\right)^{\mu/r}(\log x)^\nu\{1+o(1)\}. \tag{14}$$

We put $\gamma(n)=1$ when n is of the form (12), and $\gamma(n)=0$ otherwise. Comparing (14) with (2) and (4), we have the theorem:

If $A_\gamma(k)$ is the summatory function for the number of partitions of k into components of the form

$$n = a_0 + a_1 m + a_2 m^2 + \cdots + a_r m^r, \qquad a_r > 0,\ r \geqq 1,\ n > 0,\ m > 0,$$
$$n_1 > n_2 \text{ for } m_1 > m_2,$$

where $\sum_{m\leq m_1} 1 \sim \kappa m_1^\mu (\log m_1)^\nu$, $\mu > 0$, then

$$\log A_\gamma(k) \sim \mu^{-\mu/(r+\mu)}(r+\mu)^{(\mu-r\nu)/(r+\mu)} \cdot \left\{\kappa\Gamma\left(\frac{\mu}{r}+2\right)\zeta\left(\frac{\mu}{r}+1\right)\right\}^{r/(r+\mu)}\left\{\frac{k}{a_r}\right\}^{\mu/(r+\mu)}\{\log k\}^{r\nu/(r+\mu)}.$$

We apply this theorem to the following cases:

(A) Let m be unrestricted. Then $\kappa=1$, $\mu=1$, $\nu=0$ and

$$\log A_\gamma(k) \sim (r+1)^{1/(r+1)}\left\{\Gamma\left(\frac{1}{r}+2\right)\zeta\left(\frac{1}{r}+1\right)\right\}^{r/(r+1)}\left\{\frac{k}{a_r}\right\}^{1/(r+1)}.$$

Hardy and Ramanujan's result for partitions into rth powers of positive integers [1, p. 259], is a special case of this. (Wright [3] gives more detailed results.)

(B) Let m be restricted to primes. Then $\kappa=1$, $\mu=1$, $\nu=-1$ and

$$\log A_\gamma(k) \sim (r+1)\left\{\Gamma\left(\frac{1}{r}+2\right)\zeta\left(\frac{1}{r}+1\right)\right\}^{r/(r+1)} \cdot \left\{\frac{k}{a_r}\right\}^{1/(r+1)}\{\log k\}^{-r/(r+1)}.$$

Again, Hardy and Ramanujan's result for partitions into rth powers of primes [1, p. 260] is included as a special case.

(C) Let m be square-free. Then $\sum_{m\leq m_1} 1 \sim 6m_1/\pi^2$ and

$$\log A_\gamma(k) \sim (r+1)^{1/(r+1)}\left\{\frac{6}{\pi^2}\,\Gamma\left(\frac{1}{r}+2\right)\zeta\left(\frac{1}{r}+1\right)\right\}^{r/(r+1)}\left\{\frac{k}{a_r}\right\}^{1/(r+1)}.$$

In particular, for partitions into square-free numbers:

$$\log a_\gamma(k) \sim 2k^{1/2}.$$

(D) Let m be a polygonal number of order $q+2$ $(q \geqq 1)$:

$$m = l + q(l^2 - l)/2; \qquad l = 1, 2, 3, \cdots$$

Then

$$\sum_{m \leqq m_1} 1 \sim \left(\frac{2m_1}{q}\right)^{1/2}$$

and

$$\log A_r(k) \sim (2r+1)^{1/(2r+1)} \cdot \left\{\left(\frac{2}{q}\right)^{1/2} \Gamma\left(\frac{1}{2r}+2\right)\zeta\left(\frac{1}{2r}+1\right)\right\}^{2r/(2r+1)} \left\{\frac{k}{a_r}\right\}^{1/(2r+1)}.$$

In particular, for partitions into triangular numbers:

$$\log a_r(k) \sim \left\{\frac{3}{4}(6\pi)^{1/2}\zeta\left(\frac{3}{2}\right)\right\}^{2/3} k^{1/3}.$$

(E) Various combinations may be taken, with the answer of higher order dominating. For instance, partitions into square-free numbers and triangular numbers, with duplications omitted or not, as desired, gives the result:

$$\log a_r(k) \sim 2k^{1/2}.$$

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ON A CERTAIN WEIGHTED PARTITION FUNCTION

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1. **Introduction.** In the introduction of the preceding paper we remarked that no one has given an asymptotic formula for the number of partitions into primes. As a step in that direction we now give, under the assumption of the Riemann hypothesis, an asymptotic formula for the summatory function $A(k)$ of a certain weighted partition function $a(k)$ involving primes and powers of primes.

As pointed out in the introduction of the previous paper, the generating function of the type of partition we have in mind is given by

$$(1) \qquad g_\gamma(s) = \prod_{n=1}^{\infty} \frac{1}{(1-e^{-sn})^{\gamma(n)}} = \sum_{k=0}^{\infty} a_\gamma(k) e^{-sk},$$

where $\gamma(n) \geqq 0$, $R[s] > 0$. Using the Mellin formula for e^{-s} we get from (1)

$$(2) \qquad \log g_\gamma(s) = \frac{1}{2\pi i} \int_{3/2-\infty i}^{3/2+\infty i} \left\{ \frac{\Gamma(z)}{s^z} \sum_{n=1}^{\infty} \frac{\gamma(n)}{n^z} \sum_{m=1}^{\infty} \frac{1}{m^{z+1}} \right\} dz.$$

If $\gamma(n)$ has been chosen in such a manner as to yield an "amenable" Dirichlet series, the calculus of residues will give more or less useful information concerning $g_\gamma(s)$. For example, if $\gamma(n)=1$, then $a_\gamma(k) = p(k)$, and the method leads directly to the usual transformation formula for the generating function of the unrestricted partition function. If we consider the weighted partition function involving primes and powers of primes obtained by choosing $\gamma(n)=\Lambda(n)$, we have

$$\sum_{n=1}^{\infty} \frac{\gamma(n)}{n^z} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z} = -\frac{\zeta'(z)}{\zeta(z)},$$

and the properties of the latter function are well known. For this choice of $\gamma(n)$ we drop subscripts on $a_\gamma(k)$ and $g_\gamma(s)$ and write

$$A(k) = \sum_{n \leq k} a(n).$$

If we were to choose $\gamma(n)$ as 1 if n is a prime and 0 otherwise, then $a_\gamma(k)$ would be exactly the number of partitions of k into primes, but we would have to deal with the less tractable function $\sum p^{-z}$.

Using (2) with $\gamma(n)=\Lambda(n)$, we obtain by means of the calculus of residues an asymptotic formula for $g(s)$ as s tends to O within an

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angle of the form $|\arg s| \leqq \theta_0 < \pi/2$, namely

$$g(s) = K_2 s^{\log 2\pi} \exp\left(\frac{\pi^2}{6s} + Z(s) + O(|s|)\right), \tag{3}$$

where

$$K_2 = \exp\left(-1 + \frac{\pi^2}{24} + \sum_\rho \frac{1}{\rho^2}\right), \tag{4}$$

and

$$Z(s) = \sum_\rho \frac{\pi(2\pi)^\rho \zeta(-\rho)}{\rho s^\rho \sin(\pi\rho/2)} = -\sum_\rho \frac{\zeta(1+\rho)\Gamma(\rho)}{s^\rho}, \tag{5}$$

the summations being taken over the complex zeros of $\zeta(z)$. We shall show that $Z(s) = o(|s|^{-1})$, so that $\pi^2/6s$ is the dominant term in the exponent of e in (3). Now if we assume the Riemann hypothesis, we can show that $g(s)$ satisfies the conditions of a Tauberian theorem of Ingham [2]. Applying this theorem, we get

$$A(k) \sim K_1 \sigma^{1/2+\log 2\pi} \exp\left(\frac{1}{\sigma}\left\{\frac{\pi^2}{3} + \sigma Z(\sigma) - \sigma^2 Z'(\sigma)\right\}\right), \tag{6}$$

where $K_1 = (3/2\pi)^{1/2} K_2/\pi$, and σ is the positive solution, existent and unique for k sufficiently large, of

$$\frac{1}{\sigma^2}\left(\frac{\pi^2}{6} - \sigma^2 Z'(\sigma)\right) = k. \tag{7}$$

As k tends to infinity, σ tends to 0 and

$$\sigma Z(\sigma) \to 0, \qquad \sigma^2 Z'(\sigma) \to 0, \qquad \frac{1}{\sigma} \sim \frac{(6k)^{1/2}}{\pi},$$

in conformity with (9) of the previous paper. In contrast to (9) of the previous paper, the powers of primes make an essential contribution to (6).

In order to get an asymptotic formula for $a(k)$ we would need to apply Ingham's theorem to

$$\sum_{k=0}^{\infty} \{a(k) - a(k-1)\} e^{-sk} = (1 - e^{-s}) g(s).$$

However, the use of Ingham's theorem would require that we know that $a(k)$ is nondecreasing for large k, which seems difficult to establish (if true).

In the Hardy-Ramanujan Farey-arc method [1][1], one would consider

$$h(x) = \prod_{n=1}^{\infty} \frac{1}{(1-x^n)^{\Lambda(n)}}$$

in the neighborhood of its singularities at the points $e^{2\pi ai/b}$, where a and b are integers. In our case we have $x=e^{-s}$, $R[s]>0$, and consider the approach of s to zero. This corresponds to considering only the most important of the singularities, namely that at $x=1$.

2. **The asymptotic formula for** $g(s)$. Taking the logarithm of (1), with $\gamma(n)=\Lambda(n)$, we have

$$\log g(s) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\Lambda(n)}{m} e^{-sn^m},$$

where s is complex, $|s| \leqq 1$, $|\arg s| \leqq \theta_0 < \pi/2$, and $\log g(1)$ is real.

Using the Mellin transform,

$$e^{-s} = \frac{1}{2\pi i} \int_{3/2-\infty i}^{3/2+\infty i} \frac{\Gamma(z)}{s^z}\, dz,$$

we find

$$\log g(s) = \frac{1}{2\pi i} \int_{3/2-\infty i}^{3/2+\infty i} \frac{\Gamma(z)}{s^z} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z} \sum_{m=1}^{\infty} \frac{1}{m^{z+1}}\, dz,$$

or

$$\log g(s) = \frac{-1}{2\pi i} \int_{3/2-\infty i}^{3/2+\infty i} \frac{\Gamma(z)\zeta(z+1)}{s^z} \frac{\zeta'(z)}{\zeta(z)}\, dz. \tag{8}$$

Now we consider

$$\frac{-1}{2\pi i} \int \frac{\Gamma(z)\zeta(z+1)}{s^z} \frac{\zeta'(z)}{\zeta(z)}\, dz, \tag{9}$$

where the integral is taken around the rectangle $3/2-T_g i$, $3/2+T_g i$, $-3/2+T_g i$, $-3/2-T_g i$. T_g [3, pp. 340–341] is an ordinate between the positive integers g and $g+1$ upon which $\zeta(z)$ has no zeros, and for which

$$\left| \frac{\zeta'(x \pm T_g i)}{\zeta(x \pm T_g i)} \right| < C \log^2 T_g, \quad -\frac{3}{2} \leqq x \leqq \frac{3}{2}, \quad g \geqq 2. \tag{10}$$

[1] Numbers in brackets refer to the references cited at the end of the paper.

Note: We put $z=x+yi$, since we later wish to follow Ingham [2], who uses $s=\sigma+ti$. We then put $y=T_\nu$ for the particular set of ordinates in question, following Landau [3].

We estimate the integral over the end segments $(3/2+T_\nu i, -3/2+T_\nu i)$ and $(-3/2-T_\nu i, 3/2-T_\nu i)$. On these segments we have, in addition to (10), $|\zeta(z+1)| \leqq CT_\nu^2$ [5, p. 276]. For fixed x, and $y \to \pm\infty$, it is easy to show that (cf. [4, Satz 299])

$$|\Gamma(z)| \leqq Ce^{-\pi T_\nu/2} T_\nu.$$

Putting (s fixed)

$$s = re^{i\theta}, \qquad 0 < r \leqq 1, \qquad |\theta| \leqq \theta_0 < \pi/2, \tag{11}$$

we have $|s^{-z}| \leqq Ce^{T_\nu \theta_0}$.

Thus the absolute value of the integral (9) over the end segments is less than or equal to

$$C(e^{-\pi T_\nu/2} T_\nu)(T_\nu^2)(e^{T_\nu\theta_0})(\log^2 T_\nu) \leqq C \exp\,(T_\nu(\theta_0 - \pi/2) + 4 \log T_\nu).$$

With $\pi/2-\theta_0=\theta_1>0$, we may absorb the 4 log T_ν by using as an exponent $\theta_1 T_\nu/2$. We obtain as our final estimation,

$$Ce^{-\theta_1 T_\nu/2}, \qquad \theta_1 > 0, \qquad T_\nu > 2.$$

If we denote the sum of the residues within the rectangle of integration by Σ_{T_ν}, (9) becomes

$$\begin{aligned} &\frac{-1}{2\pi i}\int_{3/2-T_\nu i}^{3/2+T_\nu i} \frac{\Gamma(z)\zeta(z+1)}{s^z}\,\frac{\zeta'(z)}{\zeta(z)}\,dz \\ &+\frac{-1}{2\pi i}\int_{-3/2+T_\nu i}^{-3/2-T_\nu i} \frac{\Gamma(z)\zeta(z+1)}{s^z}\,\frac{\zeta'(z)}{\zeta(z)}\,dz + O(e^{-\theta_1 T_\nu/2}) = \Sigma_{T_\nu}. \end{aligned} \tag{12}$$

Next we compute Σ_{T_ν}. For this purpose we alter the integrand of (12) by means of

$$\zeta(1+z) = 2(2\pi)^z \zeta(-z) \cos\frac{\pi z}{2}\,\Gamma(-z) \qquad \text{[3, p. 285]}, \tag{13}$$

and

$$\Gamma(z)\Gamma(-z) = \frac{-\pi}{z \sin \pi z}, \qquad \text{[5, p. 239]}, \tag{14}$$

which gives

$$\frac{\pi(2\pi)^z}{z \sin \pi z/2}\cdot\frac{1}{s^z}\cdot\frac{\zeta'(z)}{\zeta(z)}\cdot\zeta(-z). \tag{15}$$

At $s=1$, $s=-1$, and at each of the complex zeros of $\zeta(s)$, (15) has a simple pole. (15) has a double pole at the origin.

We compute the residues at the simple poles first.

Since

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \cdots,$$

the residue at $s=1$ is

$$-\pi(2\pi)\frac{1}{s}\zeta(-1) = \frac{\pi^2}{6s}. \tag{16}$$

The sum of the residues at the complex zeros of $\zeta(s)$ is

$$\sum_{\rho,|\gamma|<T_\nu} \frac{\pi(2\pi)^\rho\zeta(-\rho)}{\rho(\sin(\pi\rho/2)s^\rho}, \tag{17}$$

where $\rho=\beta+\gamma i$ runs over the complex zeros of $\zeta(s)$. Each zero is counted to its multiplicity.

Since $\zeta(s)=1/(s-1)+\cdots$, the residue at $s=-1$ is

$$-\frac{1}{2}s\frac{\zeta'(-1)}{\zeta(-1)} = O(r), \qquad r\to 0. \tag{18}$$

In order to obtain the residue at the double pole at the origin, we use the following expansions ($P(z)$ denotes a power series convergent for $|z|$ small enough)

$$\left(\frac{2\pi}{s}\right)^s = 1 + s\log\frac{2\pi}{s} + s^2P(s),$$

$$\frac{1}{\sin(\pi z/2)} = \frac{2}{\pi z} + zP(z),$$

$$\zeta(s) = -\frac{1}{2} - \frac{1}{2}(\log 2\pi)s - \frac{1}{4}\left(1 - \frac{\pi^2}{24} + \log^2 2\pi - \sum_\rho \frac{1}{\rho^2}\right)z^2 + z^3P(z),$$

$$\zeta'(s) = -\frac{1}{2}\log 2\pi - \frac{1}{2}\left(1 - \frac{\pi^2}{24} + \log^2 2\pi - \sum_\rho \frac{1}{\rho^2}\right)z + z^2P(z),$$

$$\frac{1}{\zeta(z)} = -2 + 2(\log 2\pi)z + z^2P(z).$$

The expansions for $\zeta(z)$ come from

$$\zeta(z) = \frac{e^{bz}}{2(z-1)\Gamma(z/2+1)} \prod_\rho \left\{\left(1 - \frac{z}{\rho}\right)e^{z/\rho}\right\} \quad [3, \text{p. } 115],$$

and

$$\frac{1}{\Gamma(z+1)} = e^{\gamma z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)e^{-z/n}\right\},$$

where $b = \log 2\pi - 1 - \gamma/2$, γ being Euler's constant.

We find that the residue at $z=0$ is

$$\log 2\pi \log s - 1 + \frac{\pi^2}{24} + \sum_\rho \frac{1}{\rho^2}. \tag{19}$$

Inserting (16), (17), (18), and (19) in (12), and allowing $T_\rho \to \infty$, we obtain from (8)

$$\log g(s) = \frac{1}{2\pi i}\int_{-3/2+\infty i}^{-3/2-\infty i} \frac{\Gamma(z)\zeta(z+1)}{s^z}\frac{\zeta'(z)}{\zeta(z)}\,dz + \frac{\pi^2}{6s} + Z(s) + \log 2\pi \log s - 1 + \frac{\pi^2}{24} + \sum_\rho \frac{1}{\rho^2} + O(r), \tag{20}$$

where $Z(s)$ is given by (5).

In the integrand of the absolutely convergent integral which appears in (20), (11) gives $|s^{-z}| = r^{3/2}e^{-\vartheta v}$. Thus the integral is $O(r^{3/2}) = O(r)$ for r tending to zero, and so we get the transformation formula

$$\log g(s) = \frac{\pi^2}{6s} + Z(s) + \log 2\pi \log s - 1 + \frac{\pi^2}{24} + \sum_\rho \frac{1}{\rho^2} + O(r). \tag{21}$$

This gives (3) at once.

Since $1/(1-e^{-sn}) = (1/sn)\{1+o(1)\}$ as $|s| \to 0$, each factor of (1) (with $\gamma(n) = \Lambda(n)$) for which $\Lambda(n) \neq 0$ makes a contribution of importance to (3). Thus the powers of primes make an essential contribution to the transformation formula, and hence ultimately to our result for $A(k)$. The "size" of this contribution is readily seen: Write

$$g(s) = g_1(s)g_2(s) = \left(\prod_p \frac{1}{(1-e^{-sp})\log p}\right)\left(\prod_{p^m, m\geq 2} \frac{1}{(1-e^{-sp^m})\log p}\right),$$

and since $\sum_{p^m \leq x; m \geq 2} \log p \sim x^{1/2}$, (6) of the preceding paper applied to $g_1(s)$ and $g_2(s)$ separately gives

$$(22) \qquad g(s) = \exp\left(\frac{\pi^2}{6s}\{1+o(1)\} + \frac{\pi^{1/2}}{4}\zeta\left(\frac{3}{2}\right)\frac{1}{s^{1/2}}\{1+o(1)\}\right).$$

Comparison of (21) and (22) shows that $\lim_{r\to 0} rZ(r) = 0$. For future use we prove more, namely

$$(23) \qquad \lim_{r\to 0} r^{n+1}Z^{(n)}(s) = 0,$$

where $Z^{(n)}(s)$ means the nth derivative, $n \geqq 0$.

Using the first form of Z given in (5), we have, with $\rho = \beta + \gamma i$,

$$r^{n+1}\left|Z^{(n)}(s)\right| \leqq Cr^{n+1}\sum_{\rho} \frac{\left|\zeta(-\rho)\right| \cdot \left|(\rho)(\rho+1)\cdots(\rho+n-1)\right|}{\left|\rho\right| \cdot \left|s^{\rho+n}\right| \cdot \left|\sin \pi\rho/2\right|}.$$ [2]

But

$$\frac{\left|\zeta(-\rho)\right| \cdot \left|\rho(\rho+1)\cdots(\rho+n-1)\right|}{\left|\rho\right|} \leqq C\gamma^2\gamma^{n-1} = C\gamma^{n+1} \quad [5, \text{p. } 276],$$

and

$$\frac{1}{\left|s^{\rho+n}\right|} = \frac{1}{r^n}\frac{1}{\left|s^{\rho}\right|} \leqq \frac{C}{r^n}\frac{e^{\theta_0|\gamma|}}{r^{\beta}},$$

while

$$\frac{1}{\left|\sin(\pi\rho/2)\right|} \leqq Ce^{-\pi|\gamma|/2}.$$

Using $\pi/2 - \theta_0 = \theta_1$ as before, we have

$$r^{n+1}\left|Z^{(n)}(s)\right| \leqq Cr\sum_{\rho}\frac{e^{-\theta_1|\gamma|/2}}{r^{\beta}}.$$

But

$$\beta \leqq 1 - \frac{a}{\log|\gamma|}, \qquad a > 0 \qquad [3, \text{p. } 321],$$

and so

$$r^{n+1}\left|Z^{(n)}(s)\right| \leqq Cr\sum_{|\gamma|\leqq T}\frac{e^{-\theta_1|\gamma|/2}}{r^{1-a/\log|\gamma|}} + C\sum_{|\gamma|>T} e^{-\theta_1|\gamma|/2}.$$

[2] The second factor of the numerator is to be taken as 1 in case n is zero.

It is well known that the number of complex zeros of $\zeta(z)$ with imaginary part between T and $T+1$ is $O(\log T)$ and the number with imaginary part less than or equal to T is $O(T \log T)$ [3, pp. 337–338]. Hence

$$r^{n+1}\left|Z^{(n)}(s)\right|$$

$$\leqq C(T \log T)r^{a/\log T} + C\sum_{n=T}^{\infty} e^{-\theta_1 n/4}$$

$$\leqq C \exp\left(\frac{\log^2 T + \log T \log\log T + a \log r}{\log T}\right) + Ce^{-\theta_1 T/4}.$$

Put $T=-\log r$ and both terms $\to 0$ as $r\to 0$. This establishes (23).

3. **Application of Ingham's Tauberian Theorem.** We see at once that

$$g(s) = \int_0^\infty e^{-us}dA(u) = s\int_0^\infty e^{-us}A(u)du,$$

where $A(u)=A(k)$ for $k\leqq u<k+1$, and we now are in a position to apply Ingham's theorem [2]. Since it is an integral part of what follows, we quote the relevant portions of this theorem in their entirety. All statements apply for σ sufficiently small (or ω sufficiently large). Open segments: (a, b); closed segments: $[a, b]$.

We quote: "Let there be given two functions $\phi(s)$ and $\chi(s)$, and a domain D of the s-plane, satisfying the following conditions:

a.) $\phi(s)$ and $\chi(s)$ are regular in D, and real and positive on a segment $(0, h]$ of the positive real axis lying in D;

b.) $-\sigma\phi'(\sigma)$ tends monotonically to infinity as σ tends to $0 \cdots$;

c.) $\dfrac{\delta(\sigma)/\sigma}{\{\phi''(\sigma)\}^{1/2}/-\phi'(\sigma)}$ tends to ∞ as σ tends to 0,

where $\delta(\sigma)$ is the distance of the point σ from the complement of $D \cdots$;

d.) $\phi''(\sigma+z)=O\{\phi''(\sigma)\}$,

$\chi(\sigma+z)=O\{\chi(\sigma)\}$

uniformly for $|z|<\delta(\sigma)$ when σ tends monotonically to 0.

Define

$$f_0(s) = \chi(s)e^{\phi(s)}, \qquad F_0(s) = \frac{f_0(s)}{s},$$

$$A_0(\omega) = \frac{\chi(\sigma)e^{\phi(\sigma)+\omega\sigma}}{\{2\pi\sigma^2\phi''(\sigma)\}^{1/2}} = \frac{F_0(\sigma)e^{\omega\sigma}}{\{2\pi\phi''(\sigma)\}^{1/2}},$$

where $\sigma=\sigma_\omega$ is the solution [existent and unique for sufficiently large ω, by (a.) and (b.)] of

$$-\phi'(\sigma)=\omega \qquad (0<\sigma\leqq h).$$

In these circumstances we have: Suppose that the integral

$$f(s) = \int_0^\infty e^{-us}dA(u), \qquad A(0) = 0, \qquad (s = \sigma + it)$$

is convergent for $\sigma>0$, and that

(i.) $f(s)\sim f_0(s)$ when $s\to 0$ in D;

(ii.) $f(s)=O\{f_0(|s|)\}$ when $s\to 0$ in some fixed angle 'Δ' of the form $|t|\leqq\Delta\sigma$, $(0<\Delta<\infty)$;

(iii.) $A(u)$ is increasing (in the wide sense) for $u\geqq 0$.

If (ii) holds for every fixed Δ, then

$$A(u)\sim A_0(u) \quad \text{as} \quad u\to\infty.$$"

We note that Ingham's theorem has $A(0)=0$, whereas it has been convenient for us to write $A(0)=1$. This does not affect the result.

We desire to apply Ingham's theorem with

$$\chi(s) = K_2 s^{\log 2\pi}$$

and

$$\phi(s) = \frac{\pi^2}{6s} + Z(s).$$

It must therefore be established that the conditions of Ingham's theorem are satisfied by our functions.

For D we choose the region bounded by the rays $|\theta|=\pi/4$ and the arc $|s|=1$. $\delta(\sigma)$ becomes merely the distance from σ to the boundary of D. From a figure we see that:

$$\delta(\sigma) = 2^{1/2}\sigma/2, \qquad \sigma\leqq 1/2.$$

Hypothesis (a):

$\phi(s)$ and $\chi(s)$ are regular in D.

$\chi(s)$ is real and positive on the segment $(0, 1]$.

In virtue of (23) and the fact that the complex zeros of $\zeta(z)$ occur in conjugate pairs, $\phi(s)$ is also real and positive on a segment $(0, h]$ of the positive real axis for h sufficiently small.

Hypothesis (b):

$$-\sigma\phi'(\sigma) = \frac{\pi^2}{6\sigma} - \sigma Z'(\sigma),$$

$$(-\sigma\phi'(\sigma))' = \frac{-\pi^2/6 - \sigma^2 Z'(\sigma) - \sigma^3 Z''(\sigma)}{\sigma^2}.$$

Using (23), we find that the slope of $-\sigma\phi'(\sigma)$ is negative for sufficiently small σ. Hence $-\sigma\phi'(\sigma)$ tends monotonically to ∞ as $\sigma \to 0$ in the interval $(0, h]$ for sufficiently small h.

Hypothesis (c): $\delta(\sigma)/\sigma = 2^{1/2}/2$ while

$$\frac{\{\phi''(\sigma)\}^{1/2}}{-\phi'(\sigma)} = \sigma^{1/2}\frac{\{\pi^2/3 + \sigma^3 Z''(\sigma)\}^{1/2}}{\pi^2/6 - \sigma^2 Z'(\sigma)} \to 0$$

as $\sigma \to 0$ by (23).

Hypothesis (d): We see from a figure that $\sigma + z$ lies within a circle whose center is σ and whose radius is $\delta(\sigma) = ((2)^{1/2}/2)\sigma$, $\sigma < 1/2$. Thus,

$$\left(1 - \frac{(2)^{1/2}}{2}\right)\sigma < |\sigma + z| < \left(1 + \frac{(2)^{1/2}}{2}\right)\sigma \quad (\sigma \text{ small enough}). \tag{24}$$

So

$$|\chi(\sigma + z)| \leqq C\sigma^{\log 2\sigma} \leqq C\chi(\sigma) \qquad \text{uniformly in } z.$$

Putting the first inequality of (24) into $|\phi''(\sigma+z)|$ we have, from (23),

$$|\phi''(\sigma + z)| \leqq C/\sigma^3.$$

On the other hand, using (23) again, we obtain for sufficiently small σ

$$\phi''(\sigma) > \pi^2/4\sigma^3.$$

Thus: $|\phi''(\sigma+z)| < c\phi''(\sigma)$ uniformly in z.

Hypothesis (i): This has already been established. See (21).

Hypothesis (ii): We must show that

$$|g(s)| \leqq C\chi(r)e^{\phi(r)},$$

$s \to 0$, for every fixed θ_0 of (11). This is equivalent to proving that

$$\left| s^{\log 2\sigma} \exp\left(\frac{\pi^2}{6s} + Z(s)\right)\right| \leqq r^{\log 2\sigma} \exp\left(\frac{\pi^2}{6r} + Z(r) + C\right).$$

The problem thus reduces to finding a C, independent of s, such that:

$$R\left[\frac{\pi^2}{6s}+Z(s)\right] \leqq \frac{\pi^2}{6r}+R[Z(r)]+C.$$

Clearly, $R[\pi^2/6s]=\pi^2\cos\theta/6r$, see (11).

Now we analyze:

$$Z(s) = -\sum_{\rho}\frac{\zeta(1+\rho)\Gamma(\rho)}{s^{\rho}}. \tag{5}$$

In the first place,

$$s^{-\rho} = (re^{i\theta})^{-\beta-\gamma i} = \frac{e^{\gamma\theta}}{r^{\beta}}\{\cos(\beta\theta+\gamma\log r) - i\sin(\beta\theta+\gamma\log r)\}.$$

Put

$$-\zeta(1+\rho)\Gamma(\rho) = X(\rho)+iY(\rho) = X+Yi. \tag{25}$$

This gives

$$R[Z(s)] = \sum_{\rho}\frac{1}{r^{\beta}}\{Xe^{\gamma\theta}\cos(\theta\beta+\gamma\log r) + Ye^{\gamma\theta}\sin(\theta\beta+\gamma\log r)\}.$$

Thus we must show that

$$\begin{aligned}\frac{\pi^2}{6r}(1-\cos\theta)+C \geqq \sum_{\rho}\frac{1}{r^{\beta}}\{&Xe^{\gamma\theta}\cos(\theta\beta+\gamma\log r)\\ &+ Ye^{\gamma\theta}\sin(\theta\beta+\gamma\log r)\\ &- X\cos(\gamma\log r) - Y\sin(\gamma\log r)\}.\end{aligned}$$

Now $1-\cos\theta \geqq \theta^2/4$ for $|\theta| \leqq \pi/2$. If we expand $\cos(\theta\beta+\gamma\log r)$ and $\sin(\theta\beta+\gamma\log r)$, and collect terms, it will be sufficient to show that:

$$\begin{aligned}\frac{\pi^2\theta^2}{24r}+C \geqq \sum_{\rho}\frac{1}{r^{\beta}}\{&[X\cos(\gamma\log r)+Y\sin(\gamma\log r)]\cdot[e^{\gamma\theta}\cos\theta\beta-1]\\ &+[Y\cos(\gamma\log r)-X\sin(\gamma\log r)][e^{\gamma\theta}\sin\theta\beta]\}.\end{aligned}$$

But $e^{\gamma\theta}\cos\theta\beta-1=(e^{\gamma\theta}-1)\cos\theta\beta+(\cos\theta\beta-1)$, and for all real x, $|e^{x}-1| \leqq |x|e^{|x|}$. Also

$$|\cos\theta\beta-1| = 2\sin^2\frac{\theta\beta}{2} \leqq \frac{\theta^2\beta^2}{2} \leqq \frac{|\theta|\pi}{4}.$$

Thus $|e^{\gamma\theta}\cos\theta\beta-1| \leqq |\gamma\theta|e^{|\gamma\theta|}+\pi|\theta|/4$, while $|\sin\theta\beta| \leqq |\theta|\beta \leqq |\theta|$.

Hence it will be sufficient to show that

$$\frac{\pi^2\theta^2}{24r} + C \geqq \sum_{\rho} \frac{1}{r^{\beta}} \left\{ (|X| + |Y|)(|\gamma\theta| e^{|\gamma|\theta_0} + \frac{\pi|\theta|}{4}) + |\theta| e^{|\gamma|\theta_0}(|X| + |Y|) \right\}.$$

Or, with all $\beta \leqq \beta_0$, $1/2 \leqq \beta_0 \leqq 1$, since $|\gamma| > 2$ for all γ,

$$\frac{\pi^2\theta^2}{24r} + C \geqq \frac{|\theta|}{r^{\beta_0}} \sum_{\rho} 2|\gamma|(|X| + |Y|)e^{|\gamma|\theta_0}.$$

Referring to the definition of X and Y in (25), we see that the above series converges for all fixed values of θ_0, $\theta_0 < \pi/2$. The problem finally reduces, then, to finding a C_1 such that

$$\frac{\theta^2}{r} + C_1 \geqq \frac{2K_3|\theta|}{r^{\beta_0}} \qquad \text{where } K_3 \text{ is a numerical constant.} \tag{26}$$

This is equivalent to

$$\left(\frac{|\theta|}{r^{1/2}} - K_3\right)^2 \geqq 0 \qquad \text{with } \beta_0 = \frac{1}{2} \text{ and } C_1 = K_3^2.$$

The full Riemann hypothesis is essential for our method, since (26) can be written

$$\frac{1}{r^{2\beta_0-1}}\left(\frac{|\theta|}{r^{1-\beta_0}} - K_3\right)^2 + C_1 - \frac{K_3^2}{r^{2\beta_0-1}} \geqq 0. \tag{27}$$

Unless $2\beta_0 - 1 = 0$, no C_1 can be found which will satisfy (27) for all values of s simultaneously. For if $\beta_0 = 1$, we could choose $\theta = r^{1/2}$, and then find the contradiction that $(K_3^2 - (K_3 - r^{1/2})^2)/r$ is bounded. If $1/2 < \beta_0 < 1$, we could choose θ as $K_3 r^{1-\beta_0}$ for sufficiently small r, and then we would have the contradiction that $K_3^2/r^{2\beta_0-1}$ is bounded.

Hypothesis (iii): This is assured by the definition of $A(u)$.

Since the conditions of Ingham's theorem are satisfied, we have

$$A(k) \sim \frac{K_2\sigma^{\log 2\pi} \exp(\pi^2/6\sigma + Z(\sigma) + k\sigma)}{\{2\pi\sigma^2(\pi^2/3\sigma^2 + Z''(\sigma))\}^{1/2}}, \tag{28}$$

where (7) holds.

Because of (23), the actual contribution of the denominator is

$$\left((2\pi)^{1/2}\sigma \frac{\pi}{(3)^{1/2}\sigma^{3/2}}\right)^{-1}\left(1 + \frac{3\sigma^3}{\pi^2}Z''(\sigma)\right)^{-1/2} \sim \frac{(3)^{1/2}}{(2)^{1/2}} \frac{\sigma^{1/2}}{\pi^{3/2}}. \tag{29}$$

Substituting (7) and (29) into (28), we obtain our final result, (6), where

$$K_1 = \frac{(3)^{1/2}}{(2)^{1/2}\pi^{3/2}} \quad K_2 = \frac{(6)^{1/2}}{2\pi^{3/2}} \exp\left(-1 + \frac{\pi^2}{24} + \sum_{\rho} \frac{1}{\rho^2}\right),$$

and where σ is the solution, existent and unique for σ small enough, of (7). $Z(s)$ is given by (5).

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ON FAITHFUL REPRESENTATIONS OF LIE GROUPS

HARISH-CHANDRA

Let G and H be two connected Lie groups and ϕ a continuous homomorphism of H into the group of automorphisms of G. Then we define a new group $G\times_\phi H$ as follows. The elements of $G\times_\phi H$ are pairs (g, h) $(g\in G, h\in H)$ and group multiplication is defined by

$$(g_1, h_1)(g_2, h_2) = (g_1(\phi(h_1)g_2), h_1h_2).$$

Topologically $G\times_\phi H$ is taken to be just the Cartesian product of G and H. It is then easily proved that under this topology $G\times_\phi H$ is a Lie group. It is called the semidirect product of G and H under ϕ. The object of this note is to prove the following theorem.

THEOREM. *Let G be a connected, simply connected solvable Lie group and H a connected Lie group which has a faithful representation. Let ϕ be any continuous homomorphism of H into the group of automorphisms of G. Then $G\times_\phi H$ has a faithful representation.*

The special case of this theorem when H is semisimple is due to Cartan.[1]

Let R be the field of real numbers and K the field of either real or complex numbers. Let G be a connected Lie group with the Lie algebra $\mathfrak{g}$ and θ a representation of G over K of degree $\dot{d}$. Then we denote by $d\theta$ the representation of $\mathfrak{g}$ given by[2]

$$d\theta(x) = \lim_{t\to 0} \frac{\theta(\exp tX) - I}{t} \qquad (X\in\mathfrak{g})$$

where $t\in R$ and I is the unit matrix of degree $\dot{d}$. Let $GL(K, \dot{d})$ denote the group of all nonsingular matrices of degree $\dot{d}$ with coefficients in K. Any subgroup of $GL(K, \dot{d})$ will be called a linear group of degree $\dot{d}$. Let θ be the identity representation of a linear Lie group G with the Lie algebra $\mathfrak{g}$ so that $\theta(x)=x$ $(x\in G)$. Then $d\theta$ is a faithful representation of $\mathfrak{g}$ and $\exp d\theta(X)=\theta(\exp X)=\exp X$ for any $x\in\mathfrak{g}$. Hence we may identify $\mathfrak{g}$ with $d\theta(\mathfrak{g})$ under $d\theta$. $\mathfrak{g}$ can therefore be regarded as a linear Lie algebra. In particular the Lie algebra of $GL(K, \dot{d})$ then

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[1] Cartan, J. Math. Pures Appl. vol. 17 (1938) pp. 1–12. See also Malcev, C. R. (Doklady) Acad. Sci. URSS. vol. 40 (1943) pp. 87–89.

[2] For the precise definitions of the terms used in this paper see Chevalley, *Theory of Lie groups*, Princeton University Press, 1946.

consists of all matrices of degree $\dot{d}$ with coefficients in K. We denote it by $\mathfrak{gl}(K, \dot{d})$. Given any subalgebra $\mathfrak{h} \subset \mathfrak{gl}(K, \dot{d})$, by the linear Lie group generated by $\mathfrak{h}$ we mean the analytic subgroup of $GL(K, \dot{d})$ corresponding to $\mathfrak{h}$.

Let us call a matrix subtriangular if it has zeros on and below the the diagonal. First we state the following two well known lemmas.

LEMMA 1.[3] *Let $\mathfrak{N}$ be the Lie algebra of all subtriangular $\dot{d} \times \dot{d}$ matrices over K and let G be the linear Lie group generated by $\mathfrak{N}$. Then $X \to \exp X$ is a topological mapping of $\mathfrak{N}$ onto G. Also G consists of all matrices which have zeros below the diagonal and 1 everywhere on the diagonal.*

LEMMA 2.[4] *Let G be a connected, simply connected solvable Lie group. Then every analytic subgroup of G is closed and simply connected.*

From now on I adhere strictly to the notation of my paper, *Faithful representations of Lie algebras*,[5] which will be quoted as FRL.

LEMMA 3. *Let $\mathfrak{L}$, $\mathfrak{N}$, $\mathfrak{D}$ be as in Lemma* 1 *of* FRL. *We construct the faithful representation θ of $\mathfrak{L}+\mathfrak{D}$ as described there. Then for any $Y_1, \cdots, Y_s \in \mathfrak{L}$ and $D_1, \cdots, D_s \in \mathfrak{D}$,*

(1) $\exp \theta(Y_1) \cdots \exp \theta(Y_s) \neq I$ *unless* $\exp Y_1 \cdots \exp Y_s = I''$,

(2) $\exp \theta(D_1) \cdots \exp \theta(D_s) = I$ *if and only if* $\exp D_1 \cdots \exp D_s = I'$,

where I, I', and I'' are unit matrices of suitable degrees.

If we use the notation of the proof of Lemma 1 of FRL, (1) follows immediately from the fact that $\mathfrak{A}/\mathfrak{X} \cong \mathfrak{A}^*/\mathfrak{X}^*$ where $\mathfrak{X}^* = \mathfrak{X}/\mathfrak{X}_0$. Now we prove (2). Put $\omega^*(X) = (\omega(X))^*$ for $X \in \mathfrak{L}$. Then it is easily proved by induction on s that for any $D \in \mathfrak{D}$

$$\{\theta(D)\}^s \omega^*(X) = \omega^*(D^s X), \qquad s \geqq 1.$$

Hence $(\exp \theta(D))\omega^*(X) = \omega^*((\exp D)X)$. Also since $\theta(D) = d_D^*$ is a derivation of $\mathfrak{A}^* = \mathfrak{A}/\mathfrak{X}_0$, $\exp \theta(D)$ is an automorphism of $\mathfrak{A}^*$. Put $Y_{\mathfrak{X}} = (\exp D_1) \cdots (\exp D_s)X$ $(X \in \mathfrak{L})$. Then if

$$\exp \theta(D_1) \cdots \exp \theta(D_s) = I,$$

$\omega^*(Y_{\mathfrak{X}}) - \omega^*(X) = 0$ for every $X \in \mathfrak{L}$. Hence $\omega(Y_{\mathfrak{X}} - X) \in \mathfrak{X}_0 \subset \mathfrak{X}$. Therefore $\pi\omega(Y_{\mathfrak{X}} - X) = Y_{\mathfrak{X}} - X = 0$. Since this is true for every X,

[3] Birkhoff, Ann. of Math. vol. 38 (1937) pp. 526–532.

[4] Chevalley, Ann. of Math. vol. 42 (1941) pp. 668–675.

[5] Harish-Chandra, Ann. of Math. vol. 50 (1949) pp. 68–76.

$(\exp D_1) \cdots (\exp D_s) = I'$. The converse is obvious. Hence the lemma is proved.

LEMMA 4. *Let G be a connected, simply connected solvable Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{N}$ be the maximal nilpotent ideal of $\mathfrak{g}$. Then G has a faithful representation ψ such that $d\Psi(X)$ is nilpotent for every $X \in \mathfrak{N}$.*

By Corollary 1 of FRL we can find a faithful representation ρ_0 of $\mathfrak{g}$ such that $\rho_0(X)$ is nilpotent for every $X \in \mathfrak{N}$. We can therefore choose, if necessary, a new base in our representation space such that with respect to this base the matrix representing $\rho_0(X)$ is subtriangular for every $X \in \mathfrak{N}$. Now consider the factor algebra $\mathfrak{g}/\mathfrak{N}$ which is abelian and hence nilpotent. By the same corollary it follows that $\mathfrak{g}/\mathfrak{N}$ has a faithful representation by nilpotent matrices. Hence $\mathfrak{g}$ has a representation ρ_1 such that the kernel of ρ_1 is $\mathfrak{N}$ and $\rho_1(X)$ is nilpotent for all $X \in \mathfrak{g}$. We can again arrange that $\rho_1(X)$ is subtriangular for all X. Put $\rho = \rho_0 \dot{+} \rho_1$, where $\dot{+}$ denotes direct sum. Since G is simply connected, there exist representations ψ_0 and ψ_1 of G such that $d\psi_0 = \rho_0$, $d\psi_1 = \rho_1$. Put $\psi = \psi_0 \dot{+} \psi_1$. Then $d\psi = \rho$. Let N be the analytic subgroup of G corresponding to $\mathfrak{N}$. Then from Lemma 2, N is a closed invariant subgroup. Consider $\psi(N)$. It is clear that $d\psi(Y) = \rho_0(Y) \dot{+} \rho_1(Y)$ is subtriangular for all $Y \in \mathfrak{N}$. Hence from Lemmas 1 and 2 it follows that the linear Lie group $\psi(N)$ generated by $d\psi(\mathfrak{N})$ is simply connected. Since $d\psi$ is an isomorphism, $\psi(N)$ is locally isomorphic to N. Therefore since $\psi(N)$ is simply connected, ψ maps N isomorphically. Similarly we prove that $\psi_1(G)$ is simply connected. It is clear that the kernel of ψ_1 contains N. Hence ψ_1 defines a representation ψ^* of G/N given by $\psi^*(x^*) = \psi_1(x)$ where $x \to x^*$ is the natural homomorphism of G onto $G/N = G^*$. Since $d\psi^*$ is an isomorphism, the kernel of $d\psi_1 = \rho_1$ being $\mathfrak{N}$, it follows from the simple connectivity of $\psi^*(G^*) = \psi_1(G)$ that ψ^* is an isomorphism. Hence the kernel of ψ_1 is exactly N. Let D be the kernel of ψ. Then D is contained in the kernel of ψ_1 which is N. Also since ψ is faithful on N, $D \cap N = \{e\}$ where e is the unit element of N. Hence $D = \{e\}$ and ψ is a faithful representation. Also $d\psi(X)$ is nilpotent for every $X \in \mathfrak{N}$.

Now we come to the proof of the theorem. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{N}$ the maximal nilpotent ideal of $\mathfrak{g}$. By Lemma 4, G has a faithful representation. Hence we may assume that G is a linear Lie group such that every element of $\mathfrak{N}$ is nilpotent. We keep to the notation of Lemma 3 except that $\mathfrak{L}$ is replaced by $\mathfrak{g}$. Let $\mathfrak{h}$ be the Lie algebra of H. Define a homomorphism $d\tau$ of $\mathfrak{h}$ into $\mathfrak{D}$ as follows. Let

Aut (G) be the group of automorphisms of G. Then Aut (G) is a Lie group with a Lie algebra $\mathfrak{A}$. It is well known that there exists an isomorphism λ of $\mathfrak{A}$ onto $\mathfrak{D}$ such that

$$(\exp A) \exp X = \exp ((\exp \lambda(A))X)$$

for any $A \in \mathfrak{A}$ and $X \in \mathfrak{g}$. We put $d\tau = \lambda \circ d\phi$ where $d\phi$ is the homomorphism of $\mathfrak{h}$ into $\mathfrak{A}$ induced by ϕ. Then for any $P_i \in \mathfrak{h}$, $1 \leqq i \leqq r$,

$$\phi(\exp P_1 \cdots \exp P_r) \exp X = \exp ((\exp d\tau(P_1) \cdots \exp d\tau(P_r))X).$$

Let e and e' denote the identity elements of G and H respectively. Suppose $\exp P_1 \cdots \exp P_r = e'$. Then clearly

$$\begin{aligned} \exp X &= \phi(\exp P_1 \cdots \exp P_r) \exp X \\ &= \exp ((\exp d\tau(P_1) \cdots \exp d\tau(P_r))X). \end{aligned}$$

Since this is true for every $X \in \mathfrak{g}$,

$$\exp d\tau(P_1) \cdots \exp d\tau(P_r) = I'.$$

Hence we can define a representation τ of H by the rule

$$\tau(\exp P_1 \cdots \exp P_r) = \exp d\tau(P_1) \cdots \exp d\tau(P_r) \quad (P_1, \cdots, P_r \in \mathfrak{h}).$$

Put $d\psi = \theta \circ d\tau$. Then $d\psi$ is a representation of $\mathfrak{h}$. Suppose $\exp P_1 \cdots \exp P_r = e'$ $(P_1, \cdots, P_r \in \mathfrak{h})$. Then

$$\tau(\exp P_1 \cdots \exp P_r) = \exp d\tau(P_1) \cdots \exp d\tau(P_r) = I'$$

and from Lemma 3

$$\exp d\psi(P_1) \cdots \exp d\psi(P_r) = I.$$

Hence we can again define a representation ψ of H by putting

$$\psi(\exp P_1 \cdots \exp P_r) = \exp d\psi(P_1) \cdots \exp d\psi(P_r) \quad (P_1, \cdots, P_r \in \mathfrak{h}).$$

Also since G is simply connected there exists a representation χ of G such that $d\chi(X) = \theta(X)$ for every $X \in \mathfrak{g}$. Hence

$$\chi(\exp Y_1 \cdots \exp Y_r) = \exp \theta(Y_1) \cdots \exp \theta(Y_r) \quad (Y_i \in \mathfrak{G}, 1 \leqq i \leqq r).$$

From Lemma 3 it follows that χ is faithful.

Consider the mapping μ of $G \times_\phi H$ defined by $\mu(g, h) = \chi(g)\psi(h)$. We claim that μ is a representation. For any $P \in \mathfrak{h}$ and $X \in \mathfrak{g}$ consider

$$\begin{aligned} \psi(\exp P)\chi(\exp X)(\psi(\exp P))^{-1} &= \exp d\psi(P) \exp \theta(X) \exp (-d\psi(P)) \\ &= \exp \theta(D) \exp \theta(X) \exp (-\theta(D)) \end{aligned}$$

where $D = d\tau(P)$. Now for any two elements $A, B \in \mathfrak{gl}(K, \dot{d})$,

$$\exp A \exp B \exp(-A) = \exp((\exp \operatorname{ad} A)B)$$

where ad A is defined as in FRL. Since $[\theta(D), \theta(Y)] = \theta([D, Y]) = \theta(DY)$ for any $Y \in \mathfrak{g}$, it follows immediately that

$$\begin{aligned}\exp \theta(D) \exp \theta(X) \exp(-\theta(D)) &= \exp \theta((\exp D)X) \\ &= \exp \theta(\tau(\exp P)X) \\ &= \chi(\exp \tau(\exp P)X) \\ &= \chi(\phi(\exp P) \exp X).\end{aligned}$$

Since any $h \in H$ can be written in the form $\exp P_1 \cdots \exp P_r$, $P_i \in \mathfrak{h}$, $1 \leqq i \leqq r$, $r \geqq 1$, we get

$$\psi(h)\chi(\exp X)(\psi(h))^{-1} = \chi(\phi(h) \exp X).$$

Similarly since every $g \in G$ can be written as $\exp Y_1 \cdots \exp Y_r$, $Y_i \in \mathfrak{g}$, $1 \leqq i \leqq r$, $r \geqq 1$, we have

$$\psi(h)\chi(g)(\psi(h))^{-1} = \chi(\phi(h)g).$$

Therefore

$$\begin{aligned}\mu((g_1, h_1)(g_2, h_2)) &= \mu(g_1\phi(h_1)g_2, h_1h_2) = \chi(g_1\phi(h_1)g_2)\psi(h_1h_2) \\ &= \chi(g_1)\chi(\phi(h_1)g_2)\psi(h_1)\psi(h_2) \\ &= \chi(g_1)\psi(h_1)\chi(g_2)\psi(h_2) = \mu(g_1, h_1)\mu(g_2, h_2).\end{aligned}$$

Since μ is clearly a continuous mapping it is a representation of $G \times_\phi H$. By hypothesis H has a faithful representation ν_0. Define a representation ν of $G \times_\phi H$ by $\nu(g, h) = \nu_0(h)$ and put $\xi = \mu + \nu$. Suppose (g, h) belongs to the kernel of ξ. Then since $\xi(g, h) = \mu(g, h) + \nu_0(h) = \chi(g)\psi(h) + \nu_0(h)$ and since ν_0 is faithful on H, $h = e'$. Hence g belongs to the kernel of χ. But as χ is faithful on G, $g = e$. Therefore $(g, h) = (e, e')$ and ξ is faithful on $G \times_\phi H$.

COROLLARY (MALCEV).[1] *A connected solvable Lie group G has a faithful representation if and only if $G = NA$, where N is a closed, connected, simply connected invariant subgroup and A is a connected, compact abelian subgroup such that $N \cap A = \{e\}$.*

Suppose $G = NA$. For any $a \in A$ let $\phi(a)$ denote the automorphism of N given by $\phi(a)n = ana^{-1}$ $(n \in N)$. Then it is easily seen that $(n, a) \to na$ is an isomorphism of $N \times_\phi A$ onto G. Since A is compact, it has a faithful representation. Hence by the above theorem it follows immediately that G has a faithful representation.

In order to establish the converse we make use of the following lemma which follows easily from the results of Chevalley.[4]

LEMMA 5.[6] *If G is a connected solvable Lie group and N a closed, connected invariant subgroup such that G/N is compact, then there exists a compact connected abelian subgroup A of G such that $G=AN$ and $A\cap N$ is finite.*

Returning to the corollary, suppose G is linear. Since $\mathfrak{g}$ is solvable, we deduce in the usual way that every element $X\in[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}'$ is nilpotent and therefore may be assumed to be subtriangular. Therefore by Lemmas 1 and 2 the group G' generated by $\mathfrak{g}'$ is simply connected. Let $\dot{d}$ be the degree of G and G_0 the group of all matrices in $GL(K,\dot{d})$ which have zero below the diagonal and 1 everywhere on the diagonal. By Lemma 2, G' is closed in G_0. However, since G_0 is clearly closed in $GL(K,\dot{d})$, G' is closed in $GL(K,\dot{d})$ and therefore in G. Let $x\to x^*$ denote the natural homomorphism of G onto $G/G'=G^*$. Since G^* is abelian, $G^*=T^*V^*$ where T^* and V^* are connected subgroups, T^* being compact and V^* simply connected and $T^*\cap V^*=\{e^*\}$. Let N be the complete inverse image of V^* in G. Since $N/G'=V^*$ and G' are both simply connected, N is simply connected and $G/N\cong T^*$ is compact. Therefore by Lemma 5, $G=AN$ where A is compact, connected, and abelian and $A\cap N$ is finite. Let $\sigma\in A\cap N$. Then $\sigma^r=e$ for some $r\geqq 1$. Then $\sigma^*\in V^*$ and $(\sigma^*)^r=e^*$. Since V^* is simply connected and abelian, $\sigma^*=e^*$. Hence $\sigma\in A\cap G'$. Since $X\to\exp X$ is a topological mapping of $\mathfrak{g}'$ onto G', it follows that $\sigma\in G'$, $\sigma^r=e$ implies $\sigma=e$. Hence $A\cap N=\{e\}$. The corollary is therefore proved.

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[6] This lemma was pointed out to me by Dr. G. D. Mostow.

ON A PROBLEM OF E. ČECH[1]

J. NOVÁK

Let P be an abstract set and let $\mathfrak{P}$ be the system of subsets of P. An additive, single-valued set-function u taking $\mathfrak{P}$ into $\mathfrak{P}$ and satisfying the conditions $u(0)=0$ and $u(M)\supset M$ for each subset M of P is called a topology in P, more precisely, an additive topology[2] in P. The space P with the topology u is denoted by (P, u) or, briefly, P. Instead of $u(M)$ we shall write simply uM. We define:

$$u^0M = M,\ u^1M = uM,\ u^\xi M = u(u^{\xi-1}M) \text{ for isolated ordinals } \xi,$$

and

$$u^\xi M = \bigcup_{0\leq\eta<\xi} u^\eta M \text{ for non-isolated } \xi.$$

The set M is *closed* if $M=uM$. $\phi(M)$ is the least ordinal number ξ for which the set $u^\xi M$ is closed, that is, $u^\xi M=u^{\xi+1}M$. $G(P, u)$ is the set of all ordinal numbers $\phi(M)$ where $M\subset P$.

E. Čech[3] has posed the following problem: What are necessary and sufficient conditions on a set H of ordinals in order that there exist a topology u in a countable set P for which $H=G(P, u)$? V. Jarník[4] solved the generalization of the problem for a set of arbitrary cardinal number $\aleph_\rho$; his topology, however, is not an additive one, as it satisfies only the axiom of monotony. The present paper contains a solution to this problem when u is to be additive; this solution is stated in Theorem 1. The proof is carried through for a class of cardinals including $\aleph_0$.

First we have to prove the following lemma.

LEMMA. *If the space (P, u) contains a non-closed subset, then there exists a non-closed subset $A\subset P$ such that $uA=u^2A$.*

Let M be a non-closed subset. The set $uM-M$ is nonvoid: choose x in $uM-M$ and let $A=P-(x)$. By monotony of u, $A\neq uA=P$, so A

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[1] This paper was rewritten for the Bulletin; the transcriber takes full responsibility for any errors that may have been introduced by this process.

[2] Every set function f satisfies the axiom of monotony: If $M\subset N$, then $f(M)\subset f(N)$.

[3] E. Čech, *Topologické prostory*, Časopis pro Pěstování Matematiky a Fysiki vol. 66 (1937) pp. D225–D264, p. D264.

[4] V. Jarník, *Sur un problème de M. Čech*, Věstník královské české společnosti nauk (1938) pp. 1–7.

is nonclosed but uA is closed.[5]

THEOREM 1. *Let H be a set of ordinal numbers. There exists an additive topology u in a countable set P (different from $uM=M$ for all $M\subset P$) for which $H=G(P, u)$ if and only if:*[6]

1°. *H contains only countable ordinals (that is, ordinals of the first and second number classes) including 0 and 1, and*

2°. *If α and β are ordinal numbers such that $\alpha+\beta\in H$, then $\beta\in H$.*

Necessity. Obviously $0\in H$; by the lemma, 1 also is in H. Every $\phi(M)$ is countable since P can contain no uncountable ascending family of distinct sets.[7] 2° follows from the equality $u^{\xi+\beta}M=u^{\xi+\beta+1}M$, which can be rewritten as $u^{\beta}(u^{\xi}M)=u^{\beta+1}(u^{\xi}M)$. (If P is an arbitrary set of cardinal $\aleph_\rho$, the same necessity proof works with 1° changed to allow only ordinals corresponding to cardinals not greater than $\aleph_\rho$.)

Sufficiency. This proof will be carried through for cardinals $\aleph_\rho$ satisfying the following condition: If R is a set of cardinal number $\aleph_\rho$, there exists a family $\mathfrak{S}$ of cardinal number $\aleph_{\rho+1}$ whose elements are subsets P_λ of R, $\lambda<\omega_{\rho+1}$, such that each P_λ is of cardinal $\aleph_\rho$ while

(1) $\qquad P_\lambda\cap P_\mu,\quad \lambda\neq\mu,$ has cardinal less than $\aleph_\rho$.

Such cardinals do exist; in particular, taking R to be the set of rational numbers, well-ordering the irrationals, and defining P_λ to be a sequence of rationals converging to the λth irrational, shows that $\aleph_0$ is such a cardinal. Define $M\sim 0$ to mean that the cardinal of $M<\aleph_\rho$.

We assume that H is a set of ordinals satisfying 1°, generalized to allow ordinals less than $\omega_{\rho+1}$, and 2°. To topologize a set R of the given cardinal $\aleph_\rho$, assume that H is well-ordered by magnitude, $H=\alpha_0, \alpha_1, \cdots, \alpha_\lambda, \cdots$, with $\alpha_\lambda<\omega_{\rho+1}$ and $\lambda<\omega_\sigma$, $\sigma\leqq\rho+1$. Then, by 1°, $\alpha_0=0$ and $\alpha_1=1$. For each $\lambda<\omega_\sigma$ and each $\xi\leqq\alpha_\lambda$ let $P_{\lambda\xi}$ be subsets of P_λ such that

(2) $\qquad P_{\lambda 0}\subset P_{\lambda 1}\subset\cdots\subset P_{\lambda\xi}\subset\cdots\subset P_{\lambda\alpha_\lambda}$,

(3) $\qquad P_{\tau\xi}$ and $P_{\lambda\xi+1}-P_{\lambda\xi}$ are of cardinal number $\aleph_\rho$,

and

(4) $\qquad P_{\lambda\alpha_\lambda}=P_\lambda$ and $P_{\lambda\xi}=\bigcup_{0\leqq\tau<\xi} P_{\lambda\tau}$ for non-isolated ξ.

[6] By footnote 2, u is a monotone.

[7] The condition 2° is essentially the same as Jarník's condition 3 in the paper cited under footnote 4.

[7] V. Jarník defines $\phi(0)=\phi(P)=1$; therefore according to Jarník ex definitione $1\in H$.

Set $P_{\lambda-1}=0$ and assume for convenience that -1 is less than all ordinals.

For each subset M of R and each $\lambda<\omega_\sigma$ define the symbol (λM) to be the least index ξ such that $M\cap(P_\lambda-P_{\lambda\xi})\sim 0$. Then $-1\leqq(\lambda M)\leqq\alpha_\lambda$, and $(\lambda M)=-1$ if and only if $M\cap P_\lambda\sim 0$. We now prove

$$(5)\qquad (\lambda[M\cup N])=\max[(\lambda M),(\lambda N)].$$

Indeed it follows from the definition that $(\lambda A)\leqq(\lambda B)$ if $A\subset B$. Therefore $(\lambda[M\cup N])\geqq\max[(\lambda M),(\lambda N)]$. Suppose that, on the other hand, $(\lambda M)\leqq(\lambda N)$; then the set $(M\cup N)\cap(P_\lambda-P_{\lambda(\lambda N)})\sim 0$ so that $(\lambda[M\cup N])\leqq\max[(\lambda M),(\lambda N)]$.

For each $\lambda<\omega_\sigma$ define $f_\lambda(\xi)=\xi+1$ for $0\leqq\xi<\alpha_\lambda$, $f_\lambda(-1)=-1$, and $f_\lambda(\alpha_\lambda)=\alpha_\lambda$. Say that a subset M of R is of kind α if there exist a finite set of indices, $\lambda_1,\cdots,\lambda_n$, such that $M-\bigcup_{i\leqq n}P_{\lambda_i}\sim 0$; otherwise say that M is of kind β. If F is a set of kind α, put

$$F(M)=\bigcup_{i\leqq n}P_{if_i(\lambda_iM)},$$

where, for brevity, P_{λ_i} and f_{λ_i} have been replaced by P_i and f_i. If M is of kind α, define $uM=M\cup F(M)$; if M is of kind β, define $uM=R$.

If M is of kind α, the set $F(M)$ is uniquely defined. Indeed, let $\bigcup_{i\leqq n}P_{\lambda_i}\supset M\subset\bigcup_{j\leqq m}P_{\mu_j}$ and take $i\leqq n$; then $M\cap P_{\lambda_i}-\bigcup_{j\leqq m}P_{\mu_j}\sim 0$. By (1) either λ_i is a μ_j or $(\lambda_iM)=-1$ and $P_{if_i(\lambda_iM)}=0$. Hence every contribution to $F(M)$ by a λ_i is supplied by a μ_j, and conversely.

$M\cup N$ is of kind α if and only if both M and N are of kind α. Also *for sets of kind α we have* $F(M\cup N)=F(M)\cup F(N)$. Indeed, let $M\cup N-\bigcup_{i\leqq n}P_{\lambda_i}\sim 0$. By (5) the contribution, $P_{if_i(\lambda_i[M\cup N])}$, of the ith term to $F(M\cup N)$ comes either from $F(M)$ or $F(N)$. From this it follows that *u is an additive topology.*

First, u is uniquely defined for each subset of R and $M\subset uM$; also $u0=0$ since the void set is of kind α. u is additive for sets of kind α for $u(M\cup N)=F(M\cup N)\cup M\cup N=F(M)\cup F(N)\cup M\cup N=uM\cup uN$. Every set $M\sim 0$ is of kind α and $F(M)=0$ so $uM=M$ if $M\sim 0$. If either M or N is of kind β, $u[M\cup N]=R=uM\cup uN$.

Since $(\lambda P_{\lambda\xi})=\xi$, (3) and (4) imply that $uP_{\lambda\xi}=P_{\lambda\xi+1}$ for $0\leqq\xi<\alpha_\lambda$ and $uP_{\lambda\xi}=P_\lambda$ for $\xi=\alpha_\lambda$. By transfinite induction the reader can easily verify from this and (4) that

$$(6)\qquad u^\eta P_{\lambda\xi}=P_{\lambda\xi+\eta}\text{ for }\xi+\eta\leqq\alpha_\lambda;\quad u^\eta P_{\lambda\xi}=P_\lambda\text{ for }\xi+\eta\geqq\alpha_\lambda.$$

It remains to show that $G(R,u)=H$. By (6) with $\xi=0$, $\eta=\alpha_\lambda$, it follows that $\phi(P_{\lambda 0})=\alpha_\lambda$. Hence $H\subset G(R,u)$. To establish the opposite

inequality we must prove $\phi(M)\in H$ if $M\subset R$. If $uM=u^2M$, $\phi(M)=0$ or 1, so it is in H by 1°. If $uM\neq u^2M$, then M is of kind α, for $uM=R=u^2M$ if M is of kind β. We note also that if $uA=uB$ *and* $uA\neq u^2A$, from the equalities $u^2A=u^2B$, it follows that $\phi(A)=\phi(B)$. Hence for $M-\bigcup_{i\leq n}P_{\lambda_i}\sim 0$ we have

$$uM = \bigcup_{i\leq n} P_{if_i(\alpha_i M)} \cup K = u\left(\bigcup_{i\leq n} P_{i(\alpha_i M)} \cup K\right),$$

where $K\sim 0$. By the remarks above, since $uM\neq u^2M$, we have

$$\phi M = \phi\left(\bigcup_{i\leq n} P_{i(\alpha_i M)} \cup K\right).$$

By (6) and additivity of u, $\phi(M)=\max_{i\leq n}\phi(P_{i(\alpha_i M)})$ so it suffices to see that every $\phi(P_{\lambda\xi})$ is in H. However if $\eta=\phi(P_{\lambda\xi})$, then by (6) $\xi+\eta=\phi(P_{\lambda 0})=\alpha_\lambda\in H$; by 2°, $\eta\in H$. This completes the proof.

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ON THE FACTORS AND FIBERINGS OF MANIFOLDS

GAIL S. YOUNG, JR.

1. Introduction. In this note I propose to discuss first some of the properties of factors of topological manifolds, possibly having boundaries. Particular emphasis is placed on the 1- and 2-dimensional factors, which are shown to be manifolds themselves, thus generalizing slightly results of Borsuk [2].[1] These results are then applied to fiber bundles, and to the question as to whether E^n, Euclidean n-space, is a fiber bundle with a compact fiber. As a preliminary, I prove an elementary theorem which asserts that if a fiber bundle is connected and locally connected, then the components of the fiber are all homeomorphic, and which apparently has escaped notice. From this, the other theorems of this paper, and known results, follow easily several results concerning impossible fiberings of E^n. In particular it is shown that, except for possibly a finite set as fiber, there is no compact fibering of E^3 or E^4, or of E^5, one possibility excepted.

Many of the proofs are modifications of arguments due to Borsuk [2] and Montgomery and Samelson [7]. Several conversations with Samelson were helpful and stimulating.

2. 1- and 2-dimensional factors of manifolds. The first two theorems of this section are extensions of Theorems 3 and 13 of Borsuk's [2] and lean heavily on his arguments. The ingenious use of the Brouwer theorem on invariance of domain in Theorem 1, originating with Borsuk, was also discovered practically simultaneously by Szumbarski [10], and used for a similar purpose.

THEOREM 1. *If $A \times B$ is an n-manifold*[2] *(possibly with boundary) and some 0-dimensional subset of A separates an open subset of A, then A is homeomorphic to a connected subset of a simple closed curve.*

PROOF. We recall that since $A \times B$ is locally connected, so are A and

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] A connected space M is an n-manifold if every point of M has a neighborhood which is homeomorphic to Euclidean n-space E^n, or to the set of all points in E^n whose nth coordinate is non-negative. The set of all points in M not lying in neighborhoods of the first kind constitute the boundary of M. We use $A \times B$ for the topological product of two spaces.

B. Let D be a connected open subset of A, and suppose that the 0-dimensional subset T of D separates it. By a consequence of local connectivity [12; III, 1.4], we can assume that T is closed in D, so that $T\times B$ is closed in $D\times B$ and separates it. Since no subset of an n-manifold of dimension less than $n-1$ separates any open subset of it, dim $T\times B\geqq n-1$. No point of T is an interior point, so that neither is any point of $T\times B$. Hence dim $T\times B\neq n$ [5; Theorem IV, 3], and must be $n-1$. But, by [5; Theorem III, 4], dim $B\leqq$dim $T\times B$ $\leqq$dim $T+$dim $B=$dim B, so that dim $B=n-1$. If A is not homeomorphic to a subset of a circle, A contains a simple triod $ax\cup bx\cup cx$. Since ax is 1-dimensional, dim $ax\times B=n$, by [6], and hence there is a point (u, v) in $ax\times B$ interior to this set, but not lying in the boundary $\beta(A\times B)$ of the manifold $A\times B$. There is a homeomorphism h: $ax\to axb$, sending u onto x, which induces a homeomorphism h': $ax\times B\to axb\times B$, defined by $h'(p, q)=(h(p), q)$. By the Brouwer theorem on invariance of domain, $h'(u, v)=(x, v)$ is in (Int $axb\times B$) $\cap[A\times B-\beta(A\times B)]$. But (x, v) is a limit point of $(cx-x)\times B$, giving a contradiction.

THEOREM 2. *Let $A\times B$ be an n-manifold, with* dim $A=1$, 2. *Then A is a manifold, possibly with boundary, though having a boundary only if $A\times B$ does.*

PROOF. If dim $A=1$, that A is a manifold follows from Theorem 1. To show this if dim $A=2$, we need only show that A has no local cut point, and that given a point a in A, there is an open set V containing a such that every simple closed curve in V separates A, by [16; Theorem 1.1]. The first condition follows from Theorem 1, and the second condition can be obtained from Borsuk's argument for his Theorem 13, where he proves the same thing for a manifold without boundary. Only minor modifications are needed to take care of the complications introduced by the boundary.

If A has a boundary, $\beta(A)$, then $B\times\beta(A)$ is an $(n-1)$-dimensional closed subset of $A\times B$ separating no open set, which is impossible if $\beta(A\times B)$ is empty. This last remark also proves that *no manifold with boundary is a factor of a manifold without boundary*, which must be well known.

M. Szumbarski has proved in [10] that every 1-dimensional factor of a closed n-cell, I^n, is a 1-cell, and that every factor of I^n, for $1\leqq n\leqq 3$, is itself a cell. In the second of these statements, if $I^n=A\times B$, and $n\leqq 3$, then dim A, dim $B\leqq 2$ (neglecting the trivial case of A or B being a point). Hence both of these propositions are implied by the following corollary of Theorem 2.

COROLLARY 2.1. *Every* 1- *or* 2-*dimensional factor of a closed n-cell is itself a closed cell.*

PROOF. If $I^n = A \times B$, each factor is homologically trivial, by the Künneth formula. Hence if dim $A = 1$, A is a closed 1-cell, since it cannot be a 1-sphere. If dim $A = 2$, A must be a 2-manifold with boundary, since no compact 2-manifold without boundary is homologically trivial. But the only homologically trivial 2-manifold with boundary is the closed 2-cell.

Quite possibly every factor of a closed n-cell is itself a cell. However, J. H. C. Whitehead has found [11] two non-homeomorphic 2-manifolds with boundary, such that the product spaces obtained by multiplying each with an interval are homeomorphic, so that such a conjecture does not carry overwhelming conviction.

3. Factors with retraction properties. In this section I show that no manifold without boundary has as factor any set of a rather improbable type.

THEOREM 3. *If $A \times B$ is an open subset of a closed orientable n-manifold M (without boundary), and A is compact, then no nowhere dense subset of A is a deformation retract of A.*

PROOF. Suppose that the theorem is false. Then A contains a nowhere dense set R which is a deformation retract of A. If U is a small open subset of B with compact closure, then $A \times \beta(U)$ not only separates $A \times B$, but also separates M, as $A \times \overline{U}$ is compact. Now Wilder has proved [12, Theorem VII, 3.9] that if, in M, the $(n-1)$st Betti number of a closed subset T, $p_{n-1}(T)$, is k, then $M - T$ has at most $k+1$ components. Hence $p_{n-1}[A \times \beta(U)] \geqq 1$. I shall show that at least for some set U as described, $p_{n-1}[A \times \beta(U)] = 0$, thus getting a contradiction.

There is a deformation retraction r: $A \times I \to R$ satisfying $r(a, 0) = a$, $r(x, t) = x$, x in R, and $r(A, 1) = R$. For any set U, described above, this induces a deformation retraction r': $[A \times \beta(U)] \times I \to R \times \beta(U)$ by the equation $r'[(a, b), t] = [(r(a), b), t]$, a in A, b in $\beta(U)$, t in I. Then r' induces a homomorphism r': $H_q[A \times \beta(U)] \to H_q[R \times \beta(U)]$, these being homology groups, which is well known to be an isomorphism onto. Hence if dim $R \times \beta(U) < n-1$, implying $p_{n-1}[R \times \beta(U)] = 0$, the proof would be complete. However, this fact is not obvious, and an analogous statement is not true for product spaces in general. Suppose, then, that for every U in B satisfying our conditions, dim $R \times \beta(U) = n-1$. Then there exists an uncountable collection, G, of such neighborhoods, the boundaries of no two inter-

secting.[3] Each set $R\times\beta(U)$ contains a Cantorian $(n-1)$-dimensional manifold, by [5; VI 8], and which, by [1], therefore separates some open n-cell in $A\times B$. It is not hard to see that some open n-cell D in $A\times B$ is separated by uncountably many sets $[R\times\beta(U)]\cap D$. By Whyburn's [12; III, 2.2], there is an uncountable collection H of such sets forming a saturated collection; that is, given any point x in D, and a set X in H not containing x, some other set in H separates x from X in D. However, for each set $[R\times\beta(U)]\cap D$ in H, there is a point a in R which is arcwise accessible from $A-R$ by an arc T, and which is such that, for some b in $\beta(U)$, $T\times b$ is in $[A\times\beta(U)]\cap D$. It follows that each point of $T\times b-a\times b$ is separated from $[R\times\beta(U)]\cap D$ in D by some element of H. But each two sets $A\times\beta(U)$, $A\times\beta(U')$, U and U' in G, are mutually exclusive, which gives a contradiction.

Theorem 3 says, roughly speaking, that a set which is too much like a manifold with boundary cannot be a factor of a manifold without boundary, thus generalizing the statement following Theorem 2. Some light may be thrown on the proof by the following example. Pontrjagin [8] constructs two compact 2-dimensional spaces, A and B, such that $\dim A\times B=3$. Add a countable sequence of arcs, $\{T_i\}$, to A, each having only an end point in common with A, with $\bigcup T_i$ dense in A, and with $\lim \operatorname{diam} T_i=0$. Let $A'=A\cup\bigcup T_i$. Then A is a deformation retract of A', and is nowhere dense in A'. But $\dim A'\times B=\dim A\times B$.

4. **Fiber bundles.** Primarily to fix the notation, I shall state the definition of fiber bundle. A fiber bundle is the totality consisting of (a) three spaces A, B, and F, called the bundle, the base space, and the fiber, respectively; (b) a topological group G of homeomorphisms of F onto itself, called the structure group; (c) a continuous map, $p: A\to B$, called the projection, with the property that for each b in B, $p^{-1}(b)$ is homeomorphic to F; (d) a covering $\{U_\alpha\}$ of B by open sets called admissible neighborhoods, and a family of homeomorphisms ϕ_α: $U_\alpha\times F\to p^{-1}(U_\alpha)$ such that $p\phi_\alpha(b,f)=b$, for each (b,f) in $U_\alpha\times F$. These entities must also satisfy the following condition: For b fixed and each α such that U_α contains b, consider the map $\phi_{\alpha b}$: $F\to p^{-1}(b)$ defined by $\phi_{\alpha b}(f)=\phi_\alpha(b,f)$. If b is in $U_\alpha\cap U_\beta$, let $g_{\alpha\beta}=\phi_{\beta b}^{-1}\phi_{\alpha b}$. Then $g_{\alpha\beta}$ is a homeomorphism of F onto F, and we require this to be in G. It was pointed out in [3] that for G the group of all homeomorphisms of F onto F, the definition can be simplified to (a), (c), and (d). The above notation will be used without explicit mention.

[3] From this point on, the argument is a generalization of that given in my paper [14].

5. Conditions for components of a fiber to be homeomorphic. I prove several simple results, that have apparently escaped notice, concerning fibers.

THEOREM 4. *If the fiber bundle A is a connected topological space, and the fiber F is the union of two separated sets, F_1 and F_2, then some element of G maps a point of F_1 into F_2.*

PROOF. Suppose the theorem is false. Let D_i, $i=1, 2$, be the set of all points a in A such that for some U_α containing $p(a)=b$, $\phi_{\alpha b}^{-1}(a)$ is in F_i. This definition is independent of the particular choice of U_α. For, if U_β contains b, since $\phi_{\alpha b}(F_i)$ contains a, and since $\phi_{\beta b}^{-1}\phi_{\alpha b}(F_i)$ is in F_i (if for some g in G, $g(F_2)$ meets F_1, then $g^{-1}(F_1)$ meets F_2), we have $\phi_{\beta b}^{-1}(a)$ is in F_i. Clearly, $D_1 \cap D_2 = 0$. Now $U_\alpha \times F_i$, $i=1, 2$, is open in $U_\alpha \times F$, which is homeomorphic to $p^{-1}(U_\alpha)$, so $D_i \cap p^{-1}(U_\alpha)$ is open in A. But $D_i = \cup_\alpha D_i \cap p^{-1}(U_\alpha)$, so that D_i is open in A. This implies that D_1 and D_2 form a separation of A, whereas A is connected.

THEOREM 5. *If the fiber bundle A is a connected topological space, and each component of F is open in F, then each two components of F are homeomorphic, and the structure group G acts transitively on the components of F.*

PROOF. Let F_1 be the union of a component of F and all components of F homeomorphic to it, and let $F_2 = F - F_1$. Then F_1 and F_2 are mutually separated. By Theorem 4, some element g of G maps a point of F_1 into F_2. But then some component of F_1 is mapped homeomorphically onto a component of F_2 by g. This is impossible, so that $F_2 = 0$.

If the structure group G does not act transitively on the components of F, then F is the union of two separated sets F_1 and F_2 where no element of G maps a component of F_1 onto a component of F_2. This again contradicts Theorem 4.

COROLLARY 5.1. *If the fiber F has only a finite number of components, or if A is locally connected and connected, then the hypothesis on F in Theorem 5 is satisfied, and any two components of F are homeomorphic.*

The next result is suggested by the Eilenberg-Whyburn factorization theorem [12; VIII, 4.1]. For economy in space (!), I shall not state the most general hypothesis.

THEOREM 6. *Let the fiber bundle A be locally compact, metric, and connected. Let the fiber F have only a finite number of components. Let $p = lm$ be the monotone-light factorization of p, with $m(A) = C$. Then the map m defines a fibering of A with base space C, fiber F' a component of*

F, projection m and with structure group the group of all homeomorphisms of F'. The map l defines a fibering of C with base space B, fiber a set $l^{-1}(x)$, x in B, projection l, and with structure group isomorphic to the structure group G of p, modulo the subgroup of G consisting of all those elements which leave each component of F fixed; C is a covering space of B.

Notice that if we had that A were compact and locally connected, it would follow that F had only a finite number of components.

PROOF. By 5.1, all components of F are homeomorphic to one, F'. The space C is the space of all sets which are components of a set $p^{-1}(x)$, x in B, and whose open sets are defined as the set of all such components lying in an open set in A. Let U be an admissible neighborhood in B. Then $p^{-1}(U)$ is the union of a finite number of open sets, $V_1, \cdots, V_n$, each homeomorphic to $F' \times U$, and each being unions of components of sets $p^{-1}(x)$. Considering these components as points of C, the map l: $C \to B$ is 1-1 and continuous on each set V_i', where V_i' is the open set in C whose elements are subsets of V_i. It is easy to see that l is a homeomorphism on each V_i', so that C is a covering space of B, and l is a fibering of C. The statement concerning the structure group for l follows from Theorem 5.

Using the information about the sets V_i obtained from p, it is easy to see that m: $A \to C$ is a fibering as described, using the remark at the end of §4.

6. **Compact fiberings of E^n.** We are now ready to apply the above results to the question as to whether E^n, Euclidean n-space, can be a fiber bundle with a compact fiber, raised by Montgomery and Samelson in [7].

THEOREM 7. *There is no compact fibering of E^n by a fiber, one component of which is a k-sphere, or a k-torus, or a set of the type described in Theorem 3.*

PROOF. This is a consequence of Theorems 5, 6 and the fact that (a) as Montgomery and Samelson point out, Gysin's methods [4] show that E^n cannot be fibered with a sphere as fiber; (b) Eckmann, Samelson, and Whitehead [3] have proved that there is no such fibering with a k-torus as fiber; and (c) Theorem 3.

THEOREM 8. *There is no fibering of E^n, $n>1$, by a compact fiber of dimension 1 or $n-1$.*

PROOF. We may assume, from Theorem 6, that F is connected. For

dim $F=1$, the theorem is a consequence of Borsuk's result referred to above (and generalized in Theorem 2), and Theorem 7.

If dim $F=n-1$, then dim $B=1$. For if dim $B>1$, B contains uncountably many mutually exclusive compact sets $\{C_\alpha\}$ with dim $C_\alpha=1$, by [5; VII, 5]. Then by Hurewicz's theorem, dim $C_\alpha \times F=n$, so that each set $C_\alpha \times F$ has interior points. But this contradicts the separability of E^n. Hence B is either a simple closed curve or an open curve, since by Borsuk's result, it is locally Euclidean. Now if x is in B, then $p^{-1}(x)$ separates some open subset of E^n containing it, and hence separates E^n into two sets each of which is an inverse set under p^{-1}. It follows that B is not a simple closed curve. On the other hand, since $p^{-1}(x)$ is compact, one of its two complementary domains has a compact closure. Hence so does one component of $B-x$. But then B cannot be an open curve.

COROLLARY 8.1. *There is no fibering of E^2 or E^3 by a compact fiber of dimension greater than* 0.

In [15] I gave a general theorem on interior mappings of E^2 from which the part of 8.1 concerning E^2 followed. As shown there, for E^2 and the case where the dimension of the fiber is 0, results of Whyburn prove that this is also impossible. For E^n, $n>2$, the case of a compact 0-dimensional fiber is quite difficult and includes the following: If $f: E^n \to E^n$ is periodic with period k, then not every orbit of a point under f has exactly k points. (If for some such mapping, every orbit had k points, it is easy to see that E^n is a covering space of the orbit space, so that it is a fiber bundle with an orbit as fiber.)

COROLLARY 8.2. *There is no fibering of S^n, $n>1$, by a fiber of dimension $n-1$.*

PROOF. Except that B cannot conceivably be an open curve, the proof is just like Theorem 8.

THEOREM 9. *There is no compact fibering of E^n, $n>2$, by a fiber F of dimension $n-2$ onto a base space B of dimension* 2.

PROOF. Suppose again that F is connected. Since dim $B=2$, it follows from Borsuk's theorem that B is a 2-manifold. I now show that B is a plane, by showing that it is simply connected, the plane being the only non-compact simply connected 2-manifold without boundary. Let $f: I \to B$ be a closed path, I being the unit interval. Without loss, we can assume that f is constant over $[1/2, 1]$. By covering homotopy [9], and arcwise connectivity of F, there is a closed path

ϕ: $I \to E^n$ such that $p\phi = f$. Since E^n is simply connected, ϕ is homotopic to a constant mapping, so that so is f by the homotopy induced in B by p.

Let U be an admissible 2-cell neighborhood in B, and let J be a simple closed curve in U. Then J separates B into two connected sets, so that $p^{-1}(J)$ separates E^n into two connected sets. By the Alexander duality theorem, $b_{n-1}[p^{-1}(J)] = 1$, b_k denoting the kth Betti number. Since U is admissible, $p^{-1}(J)$ is homeomorphic to $J \times F$, and by the Künneth formula we must have $b_{n-2}(F) = 1$. It follows that for any point x in B, $b_{n-2}[p^{-1}(x)] = 1$, so that in $E^n - p^{-1}(x)$ there is a (singular) 1-cycle, Z^1, linked with $p^{-1}(x)$. Add a point ω_n, ω_2 to each of E^n, B to make them both spheres. Consider the singular 1-cycle $p(Z^1)$ in B. This lies in $B \cup \omega_2 - x$, an open 2-cell, so is contractible to ω_2 in this set by a homotopy $h(y, t)$. Since covering homotopy holds for compact subsets of B, we can move Z^1 into an arbitrary neighborhood of ω_n by a homotopy covering $h(y, t)$ for some range of t. This implies that Z^1 bounds in $(E^n \cup \omega_n) - p^{-1}(x)$, a contradiction.

This argument is quite closely related to Montgomery and Samelson's [7; Lemma 2]. I might note that if $b_{n-2}(F) \neq 0$, it would follow that every sufficiently small simple closed curve in B would separate it, so that B would be a 2-manifold, and the condition on dim B could be removed.

THEOREM 10. *There is no fibering of E^n by a fiber F of dimension* 2.

PROOF. Assuming F connected, it follows from Theorem 2 that it is a 2-manifold. Hence dim $B = n-2$, and we can assume dim $B > 1$, or the result follows from Theorem 8. Then B has no local cut points, so that there exists an open set U, with $\overline{U}$ in an admissible neighborhood, such that $\beta(U)$ separates B into two connected open sets. Then $b_{n-1}[F \times \beta(U)] = 1$, so that $b_2(F) = 1$. Montgomery and Samelson point out that the fundamental group of F must be abelian, and the only two orientable 2-manifolds without boundary having this property are the sphere and the torus. Hence Theorem 10 follows from Theorem 7.

COROLLARY 10.1. *There is no fibering of E^4, E^5 by a compact fiber of dimension greater than* 0, *except possibly for E^5, if* dim $F = 3$, dim $B > 2$.

PROOF. Each possible dimension of F is covered by one of Theorems 7, 8, 9, and 10.

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A PROBLEM ON INVERSE MAPPING SYSTEMS[1]

LEON HENKIN[2]

Let A be a set of indices $\alpha, \beta, \cdots$ directed by an ordering relation $<$; that is, for each α, β there is a γ such that $\alpha<\gamma$ and $\beta<\gamma$. By an *inverse-mapping-onto-system of type* A is meant a family of nonempty disjoint sets M_α together with functions Π_α^β, for each $\alpha<\beta$, mapping M_β onto M_α. It is required that these functions be such that if $\alpha<\beta<\gamma$ then $\Pi_\alpha^\beta(\Pi_\beta^\gamma x)=\Pi_\alpha^\gamma x$ for each $x\in M_\gamma$. The *inverse limit* of such a system consists of those functions f defined over A such that $f(\alpha)=x_\alpha\in M_\alpha$ and if $\alpha<\beta$ then $\Pi_\alpha^\beta x_\beta=x_\alpha$. If every inverse-mapping-onto-system of type A has a nonempty inverse limit then the directed set A is called *special*.

THEOREM. *The necessary and sufficient condition for A to be special is that it possess either a maximal element or a simple cofinal sequence.*

To show sufficiency in the case where A has a maximal element is trivial. If $\alpha_1<\alpha_2<\cdots$ is a simple cofinal sequence in A, an element of the inverse limit may be constructed by choosing x_{α_1} arbitrarily in M_{α_1}, and selecting $x_{\alpha_{i+1}}$ to satisfy $\Pi_{\alpha_i}^{\alpha_{i+1}}\ x_{\alpha_{i+1}}=x_{\alpha_i}$—possible since $\Pi_{\alpha_i}^{\alpha_{i+1}}$ is always *onto*. Finally, in case β is not one of the α_i, it is easily shown that x_β is uniquely defined by the formula $x_\beta=\Pi_\beta^{\alpha_i}x_{\alpha_i}$, where any α_i such that $\beta<\alpha_i$ is employed. (There must be one since $\{\alpha_i\}$ is cofinal.) This simple argument is contained in Tukey's paper.[1]

To show necessity we assume that A is special and construct the following inverse-mapping-onto-system of type A.

A *point* is a finite sequence $x=(\alpha_1, \alpha_2, \cdots, \alpha_{2n-1}, \alpha_{2n})$ consisting of an even number of indices from A which satisfy the following conditions:

(i) $\alpha_1<\alpha_2$,

(ii) $\alpha_{2i-1}<\alpha_{2i+2}$ for $0<i<n$,

(iii) $\alpha_{2i+1}<\alpha_{2i+2}$ and $\alpha_{2i+1}\not<\alpha_{2j+1}$ for $0\leqq j<i<n$, where $\alpha\not<\beta$ holds when neither $\alpha<\beta$ nor $\alpha=\beta$.

We define *index* $x=\alpha_{2n-1}$, *order* $x=\alpha_{2n}$, *length* $x=n$.

Let M_α consist of all points of index α. Given $\alpha<\beta$, we define the mapping Π_α^β by prescribing its image for an arbitrary point

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[1] The problem treated and notation used in this paper are taken from J. W. Tukey, *On denumerability in topology*. The theorem proved is one of two alternative conjectures made there about the problem.

[2] Frank B. Jewett Fellow.

$x=(\alpha_1, \alpha_2, \cdots, \alpha_{2n})$ of M_β (so that $\alpha_{2n-1}=\beta$) as follows. In case $\alpha \leqq \alpha_1$ let $\Pi_\alpha^\beta x=(\alpha, \alpha_2)$; since x is a point we have $\alpha_1<\alpha_2$, whence $\alpha \leqq \alpha_1$ implies $\alpha<\alpha_2$ so that (α, α_2) is a point. In case $\alpha \not< \alpha_1$ we can find (since $\alpha<\beta=\alpha_{2n-1}$) a least j such that $\alpha \leqq \alpha_{2j+1}$ (whence $\alpha \not< \alpha_{2k+1}$ for $k<j$). In that case we set $\Pi_\alpha^\beta x=(\alpha_1, \alpha_2, \cdots, \alpha_{2j}, \alpha, \alpha_{2j+2})$, which is easily seen to satisfy conditions i, ii, and iii if x does.

These sets M_α and mappings Π_α^β form an inverse-mapping-onto-system of type A. It is a simple matter to check the transitivity of the mappings: $\Pi_\alpha^\beta \Pi_\beta^\gamma=\Pi_\alpha^\gamma$ for $\alpha<\beta<\gamma$. To see that Π_α^β is onto (where $\alpha<\beta$), let $x=(\alpha_1, \alpha_2, \cdots, \alpha_{2n})$ be any point of M_α (so that $\alpha_{2n-1}=\alpha$). Choose $\gamma>\beta$ and consider the sequence $y=(\alpha_1, \alpha_2, \cdots, \alpha_{2n}, \beta, \gamma)$. Since x is a point and $\alpha_{2n-1}=\alpha<\beta<\gamma$, it is only necessary to verify that $\beta \not< \alpha_{2j+1}$, $j=0, 2, \cdots, n-1$, in order to conclude that y is a point. But this is certainly the case since $\beta \leqq \alpha_{2j+1}$ and $\alpha<\beta$ imply $\alpha_{2n-1}=\alpha<\alpha_{2j+1}$ contrary to the fact that x is a point and so satisfies iii. It is now easily seen that y is in M_β and $\Pi_\alpha^\beta y=x$.

Since we are assuming that A is special it follows that the above inverse-mapping-onto-system has a non-empty inverse limit. Let f be a function defined over A such that $f(\alpha)=x_\alpha \in M_\alpha$, where $\alpha<\beta$ implies $\Pi_\alpha^\beta x_\beta=x_\alpha$. The set of orders of points x_α is cofinal in A, since for any α we have $\alpha=\text{index } x_\alpha<\text{order } x_\alpha$ (by i and iii). Hence we can prove our theorem by showing that the orders of points x_α either possess a maximal element or form a simple sequence.

To see this last fact it is only necessary to observe that if length $x_\alpha=$length x_β then order $x_\alpha=$order x_β. For choose γ so that $\alpha<\gamma$ and $\beta<\gamma$. Hence we must have $\Pi_\alpha^\gamma x_\gamma=x_\alpha$ and $\Pi_\beta^\gamma x_\gamma=x_\beta$. But from the definition of the mappings Π it then follows that the orders of x_α and x_β are certain elements α_{2i} and α_{2j} in the sequence x_γ. Since $i=$length $x_\alpha=$length $x_\beta=j$ we have order $x_\alpha=\alpha_{2i}=\alpha_{2j}=$order x_β.

Thus if the lengths of the points x_α are unbounded there will be a simple sequence of orders β (one for each length), which is cofinal in A. In the contrary case the orders form a finite cofinal subset of A and so one of them must be maximal.

This completes the proof of our theorem.

COROLLARY I. *If A is special and B is a cofinal subset of A then B is special.*

COROLLARY II. *Without the axiom of choice we can construct, for each directed set A, an inverse-mapping-onto-system of type* A *which has a nonempty inverse limit if and only if A is special.*

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ON UPPER SEMICONTINUOUS FUNCTIONS

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1. **Introduction.** In his important paper [1] (see Bibliography) on the theory of the Lebesgue area of surfaces, L. Cesari established the following interesting result. Let X be the unit cube $0 \leqq x_j \leqq 1$, $j=1, \cdots, m$, of Euclidean m-space and let Y be the unit cube $0 \leqq y_i \leqq 1$, $i=1, \cdots, n$, of Euclidean n-space. Let $F(x_1, \cdots, x_m, y_1, \cdots, y_n)$ be a real-valued, bounded, upper semicontinuous function in the product space $X \times Y$. Then for each point $(x_1, \cdots, x_m) \in X$, F is a bounded, upper semicontinuous function of $(y_1, \cdots, y_n) \in Y$, and hence F attains, for each fixed $(x_1, \cdots, x_m) \in X$, a maximum value $M(x_1, \cdots, x_m)$, while $(y_1, \cdots, y_n)$ varies in Y. According to the theorem of Cesari, there exists then in X Borel measurable functions $y_i = \Phi_i(x_1, \cdots, x_m)$, $i=1, \cdots, n$, such that identically

$$F[x_1, \cdots, x_m, \Phi_1(x_1, \cdots, x_m), \cdots, \Phi_n(x_1, \cdots, x_m)] = M(x_1, \cdots, x_m).$$

The proof given by Cesari is essentially an induction proof. The purpose of this note is to give a direct proof for a more general result (see our Theorem 3.1) which contains the theorem of Cesari as a very special case. In this generalized result, X is any metric space and Y is any compact metric space. In this general setting, the concept of a real-valued Borel measurable function is replaced by the following concept (cf. Kuratowski [2, p. 177]). A single-valued transformation T from a metric space S_1 into a metric space S_2 is termed a Borel transformation if for every closed set $F_2 \subset S_2$ the inverse set $T^{-1}(F_2)$ is a Borel set in S_1 (and, hence, if the inverse of every Borel set in S_2 is a Borel set in S_1).

2. **Preliminaries.** Throughout this section X will denote a metric space and I will denote the unit interval $0 \leqq t \leqq 1$. Points in X will be denoted by x.

LEMMA 2.1. *In the product space $X \times I$ let E be a set which satisfies the following conditions.*

(a) *For each t_0 the set*

$$\underset{(x,t)}{E}\ [(x, t) \in E, t = t_0]$$

is a Borel set.

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(b) *For each x_0 the set*

$$E_t\,[(x_0, t) \in E]$$

is a closed interval containing 0. *Then E is a Borel set.*

PROOF. Let $r_1, r_2, \cdots$ be the set of rational numbers in I. For each pair of integers n and m we set

$$E_m^n = E_{(x,t)}\,[(x, r_n) \in E, 0 \leqq t \leqq r_n + 1/m, t \leqq 1].$$

It follows from condition (a) that each set E_m^n is a Borel set. Thus the sets

$$E_m = \sum_n E_m^n, \qquad m = 1, 2, \cdots,$$

are Borel sets. The reader will verify readily that $E = \prod_m E_m$. Therefore E is a Borel set.

LEMMA 2.2. *In the product space $X \times I$ let E be a set which satisfies the following conditions.*

(a) *For each t_0 the set*

$$E_{(x,t)}\,[(x, t) \in E, t = t_0]$$

is a Borel set.

(b) *For each x_0 the set*

$$E_t\,[(x_0, t) \in E]$$

is a closed interval containing 1. *Then E is a Borel set.*

PROOF. The proof is similar to that used in Lemma 2.1.

LEMMA 2.3. *Let $t = f(x)$ be a single-valued transformation from X into I and in the product space $X \times I$ let*

$$E_* = E_{(x,t)}\,[0 \leqq t \leqq f(x)], \qquad E^* = E_{(x,t)}\,[f(x) \leqq t \leqq 1].$$

A necessary and sufficient condition that $t = f(x)$ be a Borel transformation is that E_* be a Borel set. The same statement holds with E_* replaced by E^*.

PROOF. Assume that $t = f(x)$ is a Borel transformation. Then E_* and E^* clearly satisfy the conditions of Lemma 2.1 and Lemma 2.2 respectively. Hence E_* and E^* are Borel sets.

Assume that E_* is a Borel set. We assert that $f^{-1}(B)$ is a Borel set whenever B is a Borel set in I. By well known results it is sufficient to prove this assertion for the case where

$$B = \underset{t}{E}\,[t_0 \leqq t \leqq 1]. \tag{1}$$

Let B be a Borel set as given in (1). Set

$$X^0 = \underset{(x,t)}{E}\,[(x, t) \in X \times I, t = t_0], \qquad B_* = E_* X^0.$$

Let P: $(x, t) \to x$ be the projection of $X \times I$ onto X. From the definition of E_* and by well known properties of Borel sets we have the following facts concerning the sets B and B_*. (i) $f^{-1}(B) = P(B_*)$. (ii) Since E_* is assumed to be a Borel set, B_* is a Borel subset of X^0. (iii) Since P is a bicontinuous transformation from X^0 onto X and B_* is a Borel subset of X^0, $P(B_*)$ is a Borel set. From (i) and (iii) it follows that $f^{-1}(B)$ is a Borel set.

A similar reasoning applies if E^* is assumed to be a Borel set.

Let $g(x, t)$ be a real-valued, bounded, upper semicontinuous function defined on the product space $X \times C$ where C is a closed subset of I and C contains 0. Set

$$M(x) = \max g(x, t) \qquad \text{for } x \text{ fixed}, t \in C.$$

Since $g(x, t)$ is an upper semicontinuous function and C is compact, the set

$$E(x) = \underset{t}{E}\,[g(x, t) = M(x), x \text{ fixed}]$$

is a closed subset of C for each x. Set

$$t = f(x) = \min \tau \qquad \text{for } \tau \in E(x). \tag{2}$$

LEMMA 2.4. *Under the above conditions the transformation* $t = f(x)$ *defined in* (2) *is a Borel transformation from* X *into* C *and for each* $x \in X$

$$g[x, f(x)] = M(x). \tag{3}$$

PROOF. The relation (3) follows from the definition of $t = f(x)$ in (2). To prove that $t = f(x)$ is a Borel transformation we set

$$g_1(x, t) = \max g(x, \tau) \qquad \text{for } 0 \leqq \tau \leqq t, \tau \in C.$$

Since 0 is in C, $g_1(x, t)$ is defined on $X \times I$. It is easily shown that $M(x)$ and $g_1(x, t)$ are bounded, upper semicontinuous functions on $X \times I$ and $g_1(x, t) \leqq M(x)$. Thus the function

$$\phi(x, t) = M(x) - g(x, t) \geqq 0$$

is a Borel measurable function. Hence the set

$$E_0 = \underset{(x,t)}{E} [\phi(x, t) = 0]$$

is a Borel set. It is easily shown that the set E_0 is equal to the set

$$E^* = \underset{(x,t)}{E} [f(x) \leqq t \leqq 1].$$

Therefore, by the result in Lemma 2.3, $t=f(x)$ is a Borel transformation.

The reader can easily verify the following result on Borel transformations.

LEMMA 2.5. *Let Y be a metric space which is the product of n metric spaces $Y_1, \cdots, Y_n$ and let*

$$T: \quad y_i = \Phi_i(x), \qquad i = 1, \cdots, n, \; x \in X, \; y_i \in Y_i,$$

be a single-valued transformation from X into Y. A necessary and sufficient condition that T be a Borel transformation from X into Y is that each of the transformations $y_i=\Phi_i(x)$ be a Borel transformation from X into Y_i.

3. The main results.

THEOREM 3.1. *Let X be a metric space, let Y be a compact metric space, and let $F(x, y)$ be a real-valued, bounded, upper semicontinuous function defined on the product space $X \times Y$. Then there exists a Borel transformation*

$$y = \Phi(x) \tag{4}$$

from X into Y such that for each $x \in X$

$$F[x, \Phi(x)] = M(x), \text{ where } M(x) = \max F(x, y) \text{ for } x \text{ fixed}, \; y \in Y. \tag{5}$$

PROOF. Since Y is compact, Y is the continuous image of a closed subset C of the unit interval I: $0 \leqq t \leqq 1$, where C contains 0 (see Whyburn [3, p. 34]). Let us denote this continuous transformation by

$$y = \phi(t), \qquad t \in C.$$

Then

$$g(x, t) = F[x, \phi(t)] \tag{6}$$

is a real-valued, bounded, upper semicontinuous function defined on the product space $X \times C$ and (see (5))

$$M(x) = \max g(x, t) \qquad \text{for } x \text{ fixed}, \; t \in C.$$

The transformation $t=f(x)$ defined in Lemma 2.4 in terms of $g(xt)$ is a Borel transformation from X into C for which

$$g[x, f(x)] = M(x). \tag{7}$$

Set

$$y = \Phi(x) = \phi[f(x)]. \tag{8}$$

From (6), (7), and (8)

$$F[x, \Phi(x)] = F[x, \phi(f(x))] = g[x, f(x)] = M(x).$$

Thus the transformation defined in (8) satisfies the relation (5). Let B be a Borel set in Y. Then

$$\Phi^{-1}(B) = f^{-1}[\phi^{-1}(B)].$$

Since $y=\phi(t)$ is a continuous transformation from C onto Y, $\phi^{-1}(B)$ is a Borel subset of C, and since $t=f(x)$ is a Borel transformation from X into C, $f^{-1}[\phi^{-1}(B)]$ is a Borel set. Thus the transformation defined in (8) is a Borel transformation from X into Y which satisfies the conditions of the theorem.

If in the preceding theorem Y is the product of n metric spaces $Y_1, \cdots, Y_n$, the Borel transformation (4) can be written in the form

$$y_i = \Phi_i(x), \quad x \in X, \quad y_i \in Y_i, \qquad i = 1, \cdots, n. \tag{9}$$

By Lemma 2.5 we thus have the result:

COROLLARY. *Each transformation* $y_i=\Phi_i(x)$, $i=1, \cdots, n$, *given in* (9) *is a Borel transformation from* X *into* Y_i.

If X coincides with the unit cube $0 \leqq x_j \leqq 1$, $j=1, \cdots, m$, of Euclidean m-space and Y coincides with the unit cube $0 \leqq y_i \leqq 1$, $i=1, \cdots, n$, of Euclidean n-space, then we obtain as a special case the theorem of Cesari stated in the introduction.

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A NOTE ON PEANO SPACES

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In three-dimensional space set up a cylindrical coordinate system (r, θ, z). The Hahn-Mazurkiewicz theorem characterizes Peano spaces (locally connected metric (compact) continua) as the continuous images of the closed unit interval I on the z-axis. In this note we obtain an extension theorem for Peano spaces (henceforth called P-spaces) based upon this characterization.

We first define a dendrite L. To this end order the rationals in the interior of I into a sequence $\{(0, 0, r_i\}$. For each pair of positive integers i, j, let $L_{i,j}$ denote the closed line segment joining $(0, 0, r_i)$ and $(1/i+j, 1/i+j, r_i)$. Let $L_{-1,j}$, $L_{0,j}$ denote line segments from $(0, 0, 0)$ and $(0, 0, 1)$ parallel to and the same length as $L_{1,j}$ for every j. Then the dendrite L is defined to be the union of I and all the segments $L_{i,j}$. Let $a_{i,j}$ be the end point of $L_{i,j}$ which is not on I. We shall refer to $a_{i,j}$ as the free end of $L_{i,j}$.

Denote by D the sequence $\{(0, 0, d_i)\}$ consisting of the dyadic rational points interior to I enumerated in the usual way: $d_1=1/2$, $d_2=1/2^2$, $d_3=3/2^2$, $\cdots$. The following lemma is then easily established.

LEMMA. *Let Q^* be any finite or countable subset of I. Then there exists a homeomorphism $h(I)=I$ such that $h(0)=0$, $h(1)=1$, $h(Q^*-0-1)\subset D$.*

The result of this note may now be stated as follows.

THEOREM. *Let N be a P-space properly contained in the P-space P, and let $g(I)=N$ be any continuous transformation. Then there exists a homeomorphism $h(I)=I$ and a continuous transformation $f(L)=P$ with the following properties*:

(i) *on I, $f=gh$*

(ii) $f^{-1}(P-N)\subset L-I$.

In addition there exists a subset L^* of L, consisting of the union of I and a certain subcollection $L^*_{i,j}$ of the line segments $L_{i,j}$ such that

(iii) $f(\overline{L^*-I})=\overline{P-N}$,

and

(iv) *if $a^*_{i,j}$ is the free end of $L^*_{i,j}$ for each i, j while A^* is the union of the points $a^*_{i,j}$, then $f^{-1}(P-N)=L^*-I-A^*$.*

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PROOF. The proof will be given only for the nontrivial case in which N is nondegenerate. Let $T(I)=P$ be any continuous transformation. Then the open set $G=T^{-1}(P-N)$ is the union of a collection of open intervals $\{\omega_i\}$ having the respective left and right end points $\{q_i\}$ and $\{t_i\}$. It is convenient to assume that this sequence of intervals has been enumerated in order of decreasing diameters.

For each integer i for which a point q_i has been defined we choose a point q_i^* in $g^{-1}T(q_i)$. It does not follow that $q_i \neq q_j$ implies $q_i^* \neq q_j^*$. However, there are at most a countable number of q_i which have the same q_i^*. We denote these by $q_{i,1}, q_{i,2}, \cdots$ in order of decreasing diameters of the corresponding ω_i. Thus $q_i^*=q_{i,j}^*$ for all i. Denote by Q^* the set of all points q_i^* thus defined, and let $k(I)=I$ be the homeomorphism given by the lemma, and $h(I)=I$ the homeomorphism given by $h=k^{-1}$. For each pair of integers i, j for which a point $q_{i,j}$ has been defined, let $\omega_{i,j}$ be the component of G which has $q_{i,j}$ as its left end point. Set $d_i^*=k(q_i^*)$ for each i. Then, for each pair of integers i, j for which a point $q_{i,j}$ has been defined, denote by $L_{i,j}^*$ the line segment of L which has the point d_i^* as its foot, and the point $a_{i,j}^*$ as its free end.

The desired transformation $f(L)=P$ is now defined as follows:

(a) $f(I)=gh(I)=N$. Thus (i) is satisfied.

(b) $f(L_{i,j})=f(d_i)$ for every $L_{i,j}$ which is not an $L_{i,j}^*$.

(c) $f(L_{i,j}^*)=T(\tilde{\omega}_{i,j})$ in such a way that $f(d_i^*)$ agrees with the definition in (a), $f(a_{i,j}^*)=T(r_{i,j})$, and $f(L_{i,j}^*-d_i^*-a_{i,j}^*)=T(\omega_{i,j})$. Here, of course, $t_{i,j}$ denotes the right end point of $\omega_{i,j}$.

It is easily seen that the transformation $f(L)=P$ satisfies all of the required conditions.

In the important special case where N is a simple arc we may choose $g(I)=N$ as a homeomorphism and obtain the following corollary.

COROLLARY. *If N is a simple arc in P, then $f(L)=P$ may be so defined that $f(I)=N$ is topological.*

This theorem may be restated as a decomposition theorem for the P-space P, where each of the sets $f(L_{i,j}^*)$ is regarded as a P-space. Recent results of O. G. Harrold[1] should not be overlooked in the light of this theorem.

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[1] See O. G. Harrold, Duke Math. J. vol. 6 (1940) pp. 750–752. Also Bull. Amer. Math. Soc. vol. 48 (1942) pp. 561–566.

NONLINEAR NETWORKS. IV

R. J. DUFFIN

It may be well to point out explicitly that the networks considered in this paper and in the preceding papers are definitely not arbitrary nonlinear networks. Rather, it has been the aim to define *nonlinear networks which share the uniqueness properties of passive linear networks*. Physical considerations suggest that such networks may be aptly termed *reliable networks*.

The networks considered here consist of transformers with ferromagnetic cores (assumed to have negligible hysteresis) and Ohmic resistors, arbitrarily interconnected to a set of generators. It is found that as in the case of linear networks, under given electromotive forces from the generators, the currents flowing are uniquely determined after sufficient time has elapsed. Moreover, if the electromotive force is periodic, a unique, periodic current flow can exist.

The proof begins by integrating the network equations with respect to time. It turns out that the integrated equations have the same form as the network equations of a network containing linear capacitors and nonlinear resistors. The differential permeability of a ferromagnetic lies between positive limits; in the integrated equations this is imaged by the condition that the differential resistance lies between positive limits. The results stated are then easily deduced from theorems developed in preceding papers for networks containing such quasi-linear resistors [1, 2].[1] (These papers will hereinafter be referred to as I and III.)

In most applications of transformers the nonlinearity of the magnetization curve is an undesirable feature; however, the operation of flux gate magnetometers, magnetic voltage regulators, frequency multipliers, and magnetic amplifiers depends essentially on the nonlinearity.

Considerable space is devoted to the concepts of a transformer and of a transformer network. In §1 the transformers are restricted to be of a symmetrical type so that their properties can be determined fairly rigorously from elementary considerations. The general transformer is treated in §4 by means of the hypothesis of the magnetic network. The concept of topologically coupled networks employed there will be found quite interesting from an abstract point of view.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

1. Transformer networks. By a "transformer" is meant, in this section, a set of coils of insulated wire wound in uniform layers on a ring-shaped core of ferromagnetic material. A transformer network is a collection of transformers and generators, the various coils and generators being arbitrarily interconnected. Any conservative distribution of electric current in the network may be uniquely expressed as a superposition of cyclic currents $i_1, i_2, \cdots, i_n$ flowing in n closed circuits, provided these circuits are selected so as to be a maximal linearly independent set. A positive direction of circulation is arbitrarily assigned to each circuit. Let e_r be the sum of the electromotive forces of the generators acting in the rth circuit. By Faraday's law of induction, $e_r = dv_r/dt$ where v_r is the net magnetic flux linked with the rth circuit. This assumes, of course, that these circuits have no resistance. To express equations which relate electromotive forces and currents, it is now necessary to express the magnetic fluxes as a function of the currents.

The core of each transformer will be regarded as composed of a number of co-axial sub-rings with "small" cross-sectional diameter. By the assumed geometric symmetry it follows that the tangential magnetic field H is the same at every point of the sub-ring. The sign of H is determined by arbitrarily prescribing a direction of circulation in the sub-ring. According to Ampere's law the line integral of the tangential component of the magnetic field around a closed path is given by the net current linked with the path. (This assumes the so-called rational units.) If the circumference of the ring is s, then $sH = \sum C_k i_k$. Here C_k is the net number of times the electric circuit k links the sub-ring, the sense of linkage being determined by the right-hand rule.

In ferromagnetics the flux density, B, satisfies the relation $B = \mu(H)$, and if $dB = \mu' dH$, then it is an experimental fact that the differential permeability, μ', satisfies the relation $\mu_v \leqq \mu' \leqq A$. Here μ_v is the permeability of the vacuum. The positive constant A may be taken so large that this relation is uniform for all sub-rings, and in addition, that $A^{-1} \leqq \mu_v$. Let f be the magnetic flux flowing in the sub-ring under consideration. Then $f = a\mu(\sum C_k i_k s^{-1})$. Here a is the cross-sectional area of the ring. For convenience in notation this last relation may be written $f = U(\sum C_k i_k)$ where the function U has a derivative u lying between positive limits.

Let the sub-rings from all the transformers be placed in an ordered sequence. The variables of the sub-rings will be distinguished by superscripts. Thus the net flux linked with the rth electrical circuit is $v_r = \sum C_r^j f^j$. Introducing the relation for f^j found above gives the

desired relation

$$v_r = \sum C_r^i U^i(\sum C_k^i i_k). \tag{1}$$

Differentiating with respect to time yields

$$e_r = \sum\sum u^j C_r^j C_k^j i_k' = \sum L_{rk}' i_k'. \tag{2}$$

Here $i_k' = di_k/dt$, and the matrix L_{rk}' is defined by first summing over j; it may be called the *inductance matrix*. It is clear that L_{rk}' is symmetric; moreover, it is semi-definite, as may be seen from the following relation:

$$\sum\sum L_{rk}' i_r' i_k' = \sum u^j(\sum C_r^j i_r')^2 \geqq 0. \tag{3}$$

Relation (2) would of course remain valid if the core material were replaced by a material having a constant permeability of unity. Let the inductance matrix in this case be designated as L_{rk}; then it is clear from (3) that the following inequality holds:

$$A^{-1}\sum\sum L_{rk} i_r' i_k' \leqq \sum\sum L_{rk}' i_r' i_k' \leqq A\sum\sum L_{rk} i_r' i_k'. \tag{4}$$

The following observation is of interest in the theory of linear networks. Relation (3) shows that the inductance matrix L is semi-definite; this, of course, has long been known. The same formula shows that, conversely, *any semi-definite matrix is the inductance matrix of some network*. To see this, note that L is here expressed in the form $L = K^*K$ where K^* is the transpose of the matrix K. On the other hand, it is a relatively easy problem of matrix theory to show that any semi-definite matrix permits of such a factoring. The observation that an arbitrary matrix K can be achieved, in principle, by a suitable coil design completes the proof. (R. M. Foster has informed the writer that he also discovered this result and presented it some time ago in his classroom lectures. Foster's proof is quite simple.)

2. **Quasi-linear replacements.** The nonlinear relation (1) just obtained between the flux vector, v, and the current vector, i, is of the type which was called, in I and III, a quasi-linear replacement.[2]

[2] The following corrective paragraph should be added to the paper *Nonlinear networks*. III at the bottom of page 122:

"The one-dimensional mean value relation $V(x) - V(y) = V'x - V'y$ remains valid in n dimensions if V' is interpreted as $V_m' = \int_0 V'(i)d\theta$. Here θ is a scalar and $i = y + \theta(x-y)$. Clearly V_m' is a symmetric matrix. Moreover, integrating (iii) with respect to θ shows that V_m' satisfies (iii). Thus V_m' satisfies the conditions (ii) and (iii) on V', so in what follows the subscript m is deleted. While this notation causes some ambiguity, it is precisely the notation employed in *Nonlinear networks*. I."

Consider, then, an abstract transformation, $v = V(i)$, of an n-dimensional vector space with corresponding differential transformation, $dv = V'di$, where V' is the matrix of differential coefficients. If L is a symmetric semi-definite matrix, the transformation $V(i)$ is a quasi-linear replacement of the linear transformation Li if:

(i) $V(0) = 0$;

(ii) $V'(i)$ exists for all i as a symmetric matrix;

(iii) there is a positive constant, A, such that for all vectors i and z, $A^{-1}(Lz, z) \leqq (V'z, z) \leqq A(Lz, z)$.

In the last line the standard notation for the inner product has been employed; that is, if x and y are two vectors, then $(x, y) = \sum x_k y_k$. The following simple consequence will be needed later:

LEMMA. *If $V(i)$ is a quasi-linear replacement of Li, then so also is $V_1(i) = V(i+k) - V(k)$ where k is a constant vector.*

To indicate the nature of a quasi-linear transformation, several of its properties will now be mentioned: If M is the manifold determined by vectors of the form Li, and if k is any vector perpendicular to this manifold, then $V(i+k) = V(i)$. All vectors of the form $V(i)$ are contained in M. If v is in M, the relation $v = V(i)$ has a unique solution for i in M, $i = W(v)$. Moreover, the domain of W may be extended so that it also is a quasi-linear replacement of Li. Proofs of these statements are not difficult and are contained essentially in I and III.

3. **Transformer resistor networks.** To obtain the equations defining the flow of current in a general network containing linear transformers, linear resistors, and linear capacitors, it is simply necessary to add voltage drops due to the resistors and capacitors to the equation already obtained for the inductive voltage drops. It is convenient to introduce the vector y of electric charge, which is such that $i = y'$. Then the general network equations are

$$Ly'' + Ry' + Sy = e. \tag{5}$$

Here R is the matrix of resistors, its diagonal elements, say R_{kk}, being the total resistance in the kth circuit. The off-diagonal, R_{kr}, has magnitude equal to the common resistance of the circuits k and r, and its sign depends on the relative directions assigned to the circuits r and k. The formalism developed in §1 may be adapted to prove these and other properties of R. Interpret C_r^j as the incidence matrix between the branches j and the circuits r of the graph. Thus, $C_r^j = 1$ if the branch j occurs in the rth circuit and their senses agree, and $C_r^j = -1$ if their senses disagree. If the branch j does not occur in the

rth circuit, $C_r^j = 0$. In relation (1) interpret v_r as the net voltage drop across the resistors of the rth circuit, and interpret U^j as the resistance of the jth branch. Then it may be seen from relation (3) that R is semi-definite. It also follows from (3) that if none of the branches have zero resistance, R is definite. For quasi-linear resistors, U^j is interpreted as the voltage current function, and the same proof shows that a quasi-linear replacement arises.

It is not true that every symmetric semi-definite matrix is a resistor matrix of some network. To state in algebraic terms the restrictions on R to be sure that such a matrix is actually a representation of a network is a topological problem of graphs which has not been solved. The matrix S relating to the voltage drops across the capacitors has exactly the same restrictions and properties as that of R.

It is to be noted that the assumption that one or more of the matrices L, R, and S could be singular is an idealization. Thus all actual conductors have a nonvanishing resistance, and if this were assumed, it would immediately follow that the matrix R is nonsingular. It has been customary, however, and useful, to assume that some circuits contain, for example, only resistance or only capacitance, and so on. Where this idealization stops is partly a matter of taste, but the weakest assumption which appears to be useful is that the matrix $L+R+S$ is nonsingular.

The nonlinear networks considered previously were assumed to satisfy the equation

$$Ly'' + V(y') + Sy = e \tag{6}$$

where $V(y')$ was taken as a quasi-linear replacement of Ry'. The networks to be discussed here are such that $S=0$. In the linear case the equation satisfied is

$$Li' + Ri = e. \tag{7}$$

In the nonlinear case the equation satisfied is

$$dV(i)/dt + Ri = e \tag{8}$$

where $V(i)$ is a quasi-linear replacement of Li. In order to apply the theory developed in I and III, it is necessary to reduce equation (8) to that of the form of equation (6). Integrating equation (8) gives

$$V(y') + Ry = p. \tag{9}$$

Here $p = \int_0^t e\,dt + h$ where h is some constant vector of integration. Clearly, (9) is of the form (6) with $L=0$ and $S=R$.

THEOREM 1. *Let R be an arbitrary symmetric definite matrix. Let $e(t)$ be a finite integrable function for $t \geqq 0$. Then if i_1 and i_2 are two continuous solutions of equation* (8),

$$\int_0^\infty \|i_1 - i_2\|^2 dt < \infty.$$

PROOF. The hypotheses given are sufficient to insure that relation (9) is valid. Moreover, it may be assumed that $h=0$, because since R is nonsingular, there is a constant vector k such that $Rk=h$. Replacing y by $y+k$ achieves this result without changing the value of y'. Consider now the equation

$$Ly' + Ry = 0. \tag{10}$$

In order to apply the main theorem of I, it is necessary to know that all solutions of (10) are such that $y' \to 0$ as $t \to \infty$. To show this, note that in a rotated coordinate system R is a diagonal matrix with positive diagonal elements. Now by making a simple transformation which merely shortens or lengthens the coordinate axes, R can be made to be the identity matrix, E. Another rotation of axes converts L to a diagonal matrix. Thus (10) breaks up into n equations of the form $l_k y_k' + y_k = 0$, where $l_k \geqq 0$. Clearly, then, $y_k' \to 0$ as $t \to \infty$. The transformations made in going to this new coordinate system were all nonsingular, so it follows that $y_k' \to 0$ in the old coordinate system also. Thus the theorem of I is now directly applicable and completes the proof.

THEOREM 2. *Let R be an arbitrary symmetric definite matrix. Let $e(t)$ be a continuous function of period 2π. Then equation* (8) *has one and only one solution for which i is a continuous periodic function.*

PROOF. It is possible to write $e=e_1+s$ where the integral of e_1 over an interval of length 2π vanishes and where s is a constant vector. Since R is nonsingular, there is a constant vector k such that $Rk=s$. Let $i=i_1+k$. Then equation (8) can be written as $dV_1(i_1)/dt + Ri_1 = e_1$ where V_1 is the quasi-linear replacement as given by the preceding lemma. Integrating this equation puts it in the form (9), where p is a periodic function with a bounded first derivative. Theorem 5 of III is now directly applicable, and it states that there is a continuous periodic solution i_1. Theorem 1, above, implies that there is no other continuous solution of period 2π or, for that matter, of any other period. This completes the proof of Theorem 2.

4. **Interlinked electric and magnetic networks.** The roman num-

eral III depicts a commonly used transformer core shape. Apparently the analysis given in §1 is not applicable for such unsymmetrical transformers. By making use of the familiar concept of a magnetic network, the conclusions of §3 will now be shown to remain valid, at the expense, however, of introducing an additional assumption.

A magnetic network is formally analogous to a direct current electric network. First a maximal set of independent circuits is selected in the transformer cores. Then any distribution of flux in the cores can be expressed uniquely as a superposition of cyclic fluxes f^j flowing in these circuits. This statement rests on the fact that div $B=0$. In the linear case the specific magnetic resistance is taken to be $1/\mu$. In the nonlinear case the specific resistance function is the inverse of the permeability function. The magnetic network equations may then be written in analogy to the electric case. Expressed in vectorial form, they are:

$$m = M(f). \tag{11}$$

Here m may be called the *magnetomotive force vector*. By definition, the component m^j is the line integral of the tangential component of the magnetic field around the jth circuit.

Suppose now that this magnetic network is topologically interlinked with an electric network. It is desired to relate the vectors m and f of "magnetic space" with the vectors v and i of "electric space." Introducing the *linkage matrix*, C, in exact analogy to the matrix C in §1, the generalization of relation (1) is:

$$v = Cf. \tag{12}$$

On the other hand, Ampere's line integral law demands that

$$m = C^*i. \tag{13}$$

Equations (11), (12), and (13) define an implicit relation between v and i. By this method of analysis VerPlanck and Fishman [3] are able to obtain explicit solutions of several magnetic amplifier networks in terms of the basic parameters.

If the magnetic network is constructed of ferromagnetic material, the magnetic resistors may be termed quasi-linear resistors, because the differential resistance will lie between positive limits. The considerations given in §3 show that M is a quasi-linear replacement. Since none of the branches have zero resistance, it follows that M has an inverse, U. Thus

$$v = CU(C^*i). \tag{14}$$

The differential transformation is $dv = CU'C^*di$. But U is a quasi-linear replacement, so it is easy to see that the matrix $CU'C^*$ satisfies conditions (ii) and (iii). Thus (14) defines a quasi-linear replacement in electric space. The theorems in §3 are now valid, so it may be said that networks of unsymmetrical transformers and Ohmic resistors are reliable.

The analysis in this section tacitly assumes that it is permissible to treat distributed parameters as mean lumped parameters. A justification of this assumption will be attempted in a later paper. This will involve rather lengthy considerations, because the solution of a non-linear Dirichlet problem is demanded.

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"CHARACTERISTIC DIRECTIONS" IN THREE-DIMENSIONAL SUPERSONIC FLOWS

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1. **Introduction.** It is well known from H. Lewy's work [1][1] that a hyperbolic quasi-linear second order partial differential equation in two independent variables can be replaced by a canonical system of first order partial differential equations with the aid of the two families of characteristic curves. Recently, R. Courant and K. O. Friedrichs showed [2; 3] that a system of n-totally hyperbolic quasi-linear first order partial differential equations in two independent variables can be replaced by another system of such equations with the property that each of the latter equations involves differentiation in only one direction. The n-directions which possess this property are called "characteristic directions" by Courant and Friedrichs. The "characteristic directions" of this type are used by these authors in formulating a finite difference method for treating problems such as steady, non-isentropic, two-dimensional supersonic flows.

It will be shown that no such "characteristic directions" exist for the following steady, three-dimensional supersonic flows: (1) non-isentropic, (2) isentropic, irrotational. In other words, any linear combinations of the governing first order equations for these flows must involve derivatives with respect to (at least) two directions. Further, two such directions are sufficient for the second type of flow. It is shown that any two directions lying in the planes which envelope the bicharacteristic cone are permissible.

2. **The flow equations—"characteristic directions."** We shall use tensor notation although the x^λ ($\lambda=1, 2, 3$) coordinate system will be considered to be Euclidean orthogonal. The components of the velocity vector are denoted by $v^\lambda(\lambda=1, 2, 3)$; the metric tensor by $g^{\lambda\mu}$; the permutation tensor density by $\epsilon^{\lambda\mu\beta}$. Further, the pressure, p, will be considered to be a known function of the density, ρ, and the entropy, s. Hence, we shall introduce the quantities c^2, b defined by

$$c^2 = \frac{\partial p}{\partial \rho}, \qquad b = \frac{\partial p}{\partial s}. \tag{2.1}$$

For steady motion, the equation of conservation of mass, the laws of

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[1] Numbers in brackets refer to the references cited at the end of the paper.

motions, and the condition that the entropy of a given gas particle be constant are

$$\rho g^{\lambda\mu}\frac{\partial v_\mu}{\partial x^\lambda}+v^\mu\frac{\partial \rho}{\partial x^\mu}=0, \tag{2.2}$$

$$v^\lambda\frac{\partial v_\mu}{\partial x^\lambda}+\frac{c^2}{\rho}\frac{\partial \rho}{\partial x^\mu}+\frac{b}{\rho}\frac{\partial s}{\partial x^\mu}=0, \tag{2.3}$$

$$v^\mu\frac{\partial s}{\partial x^\mu}=0. \tag{2.4}$$

The conditions for irrotational motion are

$$\epsilon^{\alpha\lambda\mu}\frac{\partial v_\lambda}{\partial x^\mu}=0. \tag{2.5}$$

If we multiply the equation (2.2) by one, (2.3), (2.5) by the undetermined vectors l^λ, w_α respectively, and (2.4) by an undetermined scalar σ, we obtain, upon adding the resulting equations,

$$\begin{aligned}(\rho g^{\lambda\mu}+l^\mu v^\lambda+w_\alpha\epsilon^{\alpha\lambda\mu})\frac{\partial v_\mu}{\partial x^\lambda}\\ +\left(v^\mu+\frac{c^2}{\rho}l^\mu\right)\frac{\partial \rho}{\partial x^\mu}+\left(\frac{bl^\mu}{\rho}+\sigma v^\mu\right)\frac{\partial s}{\partial x^\mu}=0.\end{aligned} \tag{2.6}$$

For non-isentropic flows, $w_\alpha=0$, since (2.5) cannot be used; for isentropic, irrotational flows, $b=0$, $\sigma=0$, since (2.4) is identically satisfied. With this understanding, (2.6) represents the most general linear combination of (2.2), (2.3), (2.4), (2.5). Let i^λ denote a tangent vector to a congruence of space curves. That is,

$$i^\lambda=\frac{dx^\lambda}{ds}, \tag{2.7}$$

where s is some parameter along any curve of the congruence. The condition that (2.6) involve differentiation with respect to s leads to the equations

$$\rho g^{\lambda\mu}+l^\mu v^\lambda+w_\alpha\epsilon^{\alpha\lambda\mu}=p^\mu i^\lambda, \tag{2.8}$$

$$v^\mu+\frac{c^2}{\rho}l^\mu=ki^\mu, \tag{2.9}$$

$$\sigma v^\mu+\frac{bl^\mu}{\rho}=mi^\mu, \tag{2.10}$$

where p^μ is an additional undetermined vector and k, m are additional undetermined scalars. For non-isentropic flows ($w_\alpha=0$), we solve (2.9) for l^μ and substitute this result into (2.8). This leads us to the equation

$$\frac{\rho}{c^2}(c^2 g^{\lambda\mu} - v^\lambda v^\mu) + \frac{k\rho}{c^2} i^\mu v^\lambda = p^\mu i^\lambda. \tag{2.11}$$

Let

$$i^\lambda_j \qquad (j = 1, 2)$$

denote two independent vector fields orthogonal to i^λ. Forming the scalar product of (2.11) with

$$i^\lambda_k i^\mu_j,$$

we obtain

$$c^2 i^\lambda_j i^\lambda_k = (v^\lambda i^\lambda_j)(v^\lambda i^\lambda_k), \qquad j, k = 1, 2. \tag{2.12}$$

Since

$$i^\lambda_j$$

consists of two independent vector fields, equation (2.12) implies a contradiction. Hence, we conclude that *"characteristic directions" do not exist for non-isentropic, three-dimensional flows.* For isentropic, irrotational flows, $b=\sigma=0$ but w_α need not vanish. As in the previous case, we solve (2.9) for l^μ and substitute the result into (2.8). We obtain

$$\frac{\rho}{c^2}(c^2 g^{\lambda\mu} - v^\lambda v^\mu) + \frac{\rho k}{c^2} i^\lambda v^\mu + w_\alpha \epsilon^{\alpha\lambda\mu} = p^\mu i^\lambda. \tag{2.13}$$

Forming the scalar product of (2.13) with

$$i^\mu_j i^\lambda_j,$$

we find

$$c^2 = (v^\mu i^\mu_j)^2, \qquad j = 1, 2. \tag{2.14}$$

The equation (2.14) states that

$$i^\lambda_j, \qquad (j = 1, 2)$$

belong to the characteristic normal cone. Since

$$i^\lambda \\ j$$

may be taken to be any two distinct vectors in the plane perpendicular to i^λ at any point P, the normal characteristic cone must degenerate to a plane at any such point. But this implies a contradiction since the normal cone for isentropic, irrotational flows is a right circular cone with axis along the velocity vector. Thus, *"characteristic directions" do not exist for isentropic, irrotational three-dimensional flows.*

3. **"Characteristic pairs of directions."** In this section, we consider isentropic, irrotational three-dimensional flows. If two distinct congruences of curves with tangent vectors i^λ, j^λ exist such that linear combinations of the governing partial differential flow equations can be expressed as linear combinations of derivatives along these two congruences, then we say that a "characteristic pair of directions" exists. Some simplicity is obtained by eliminating ρ between (2.2), (2.3), where $b=0$ since $s=$constant throughout the flow field. Thus, (2.2), (2.3) may be replaced by

$$a^{\lambda\mu}\frac{\partial v_\lambda}{\partial x^\mu} = 0, \qquad a^{\lambda\mu} = c^2 g^{\lambda\mu} - v^\lambda v^\mu. \tag{3.1}$$

Our problem reduces to the question of determining when three vector fields w_α, p_α, q_α and two congruences of curves i_α, j_α exist such that

$$a^{\lambda\mu} + w_\alpha \epsilon^{\alpha\lambda\mu} = p^\lambda i^\mu + q^\lambda j^\mu. \tag{3.2}$$

If k^μ represents a vector field orthogonal to i^μ, j^μ and r^μ some unknown vector field, then the tensor $a^{\lambda\mu}+w_\alpha\epsilon^{\alpha\lambda\mu}$ can always be expressed in the form

$$a^{\lambda\mu} + w_\alpha \epsilon^{\alpha\lambda\mu} = p^\lambda i^\mu + q^\lambda j^\mu + r^\lambda k^\mu. \tag{3.3}$$

It is easily verified that

$$r^\lambda = (a^{\lambda\mu} + w_\alpha \epsilon^{\alpha\lambda\mu}) k_\mu. \tag{3.4}$$

Similar expressions involving i^μ, j^μ (instead of k^μ) are valid for p^λ, q^λ (when i^λ, j^λ are orthogonal). The necessary and sufficient condition that (3.3) reduce to (3.2) is that $r^\lambda=0$. That is,

$$w_\alpha(\epsilon^{\alpha\lambda\mu}k_\mu) = -\, a^{\lambda\mu}k_\mu. \tag{3.5}$$

Since $\epsilon^{\alpha\lambda\mu}k_\mu$ is an antisymmetric tensor, the rank of this tensor is

equal to zero or two. If k_μ is not identically zero, the rank is two. Evidently, (3.5) will possess ∞^1 solutions for w_α if and only if the rank of the augmented matrix of (3.5) is two. Direct computation or use of the $\epsilon^{\alpha\lambda\mu}$ symbols shows that

$$a^{\lambda\mu}k_\lambda k_\mu = 0 \tag{3.6}$$

is the necessary and sufficient condition for this to be valid. But (3.6) implies that k_μ lies in the normal characteristic cone. Hence i^λ, j^λ *must lie in a plane tangent to the bicharacteristic cone.* It is evident that if (3.6) is satisfied, then ∞^1 vectors w_α can be obtained satisfying (3.5). For any such vector w_α, equations similar to (3.4), for appropriately chosen i^λ, j^λ, determine p^λ, q^λ so that (3.2) is satisfied.

The above results furnish some additional insight into the method used by C. L. Dolph and the author in their study of three-dimensional supersonic flows. In their study, it was found that a characteristic system can be constructed from the geometry of the characteristic surfaces. However, the equations of the system contain derivatives with respect to two directions. This is essentially different from the two-dimensional case, where only one directional derivative enters into each equation of the characteristic system. But in view of the above results for three-dimensional flows, it is seen that no smaller number of derivatives than two can enter.

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SOME THEOREMS ON MEROMORPHIC FUNCTIONS

YOSHITOMO OKADA

Let $f(z)$ be an integral function of finite order $\rho \geqq 0$, and let $n(r)$ denote the zeros of $f(z)$ in $|z| \leqq r$. Set

$$M(r, f) = \max_{|z| \leqq r} |f(z)|.$$

Pólya[1] proved

$$\liminf_{r \to \infty} \frac{\log M(r, f)}{n(r)} < \infty$$

if ρ is not an integer, and Shah[2] proved

$$\liminf_{r \to \infty} \frac{\log M(r, f)}{n(r)\phi(r)} = 0$$

if $f(z)$ is a canonical product of genus p such as $\rho = p$, and $\phi(x)$ is any positive continuous nondecreasing function of a real variable x such that $\int_a^\infty dx/x\phi(x)$ is convergent.

In this paper, some similar theorems will be obtained for meromorphic functions.

Let

$$F(z) = f(z)/g(z) \tag{1}$$

be a meromorphic function of finite order, where

$$f(z) = \prod_{n=1}^{\infty} E\left(\frac{z}{a_n}, p\right)$$

$$= \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left(\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{p}\left(\frac{z}{a_n}\right)^p\right),$$

$$g(z) = \prod_{n=1}^{\infty} E\left(\frac{z}{b_n}, q\right)$$

$$= \prod_{n=1}^{\infty} \left(1 - \frac{z}{b_n}\right) \exp\left(\frac{z}{b_n} + \frac{1}{2}\left(\frac{z}{b_n}\right)^2 + \cdots + \frac{1}{q}\left(\frac{z}{b_n}\right)^q\right),$$

these being canonical products of genera p, q and of orders ρ_1, ρ_2, respectively. Let s be the genus of (1), that is $s = \max(p, q)$, and de-

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[1] G. Pólya, Math. Ann. vol. 88 (1922) pp. 169–183.

[2] S. H. Shah, J. London Math. Soc. vol. 15 (1940) pp. 23–31.

note the number of the totality of $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ in $|z| \leqq r$ by $N(r)$. Moreover set

$$M(r, F) = M(r) = \max_{|z| \leqq r} |F(z)|.$$

Then we have the following theorems:

THEOREM I. *If for any given meromorphic function* (1) *of finite order*

$$\max(\rho_1, \rho_2) = s \geqq 0,$$

then

$$\liminf_{r \to \infty} \frac{1}{rN(r)\phi(r)} \int_0^r \log^+ M(t)dt = 0, \tag{2}$$

where $\phi(x)$ is any positive continuous nondecreasing function of a real variable x such that $\int_a^\infty dx/x\phi(x)$ is convergent.

THEOREM II. *If for any given meromorphic function* (1) *of finite order* $\max(\rho_1, \rho_2)$ *is not an integer, then*

$$\liminf_{r \to \infty} \frac{1}{rN(r)} \int_0^r \log^+ M(t)dt < \infty. \tag{3}$$

To prove these theorems we establish the lemma:

LEMMA. *For any given meromorphic function* (1) *of genus s, let $c_1, c_2, \cdots$ denote the totality of $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ in $|z| \leqq r$. Then*

$$\frac{1}{r} \int_0^r \log^+ M(t)dt \leqq h_1 \sum_{n=1}^{\infty} \frac{r^{s+1}}{|c_n|^s(|c_n| + r)}, \tag{4}$$

when $s \geqq 1$; and for a number $k > 1$,

$$\frac{1}{r} \int_0^r \log^+ M(t)dt \leqq h_2 \sum_{n=1}^{\infty} \log\left(1 + \frac{kr}{|c_n|}\right), \tag{5}$$

when $s = 0$.

PROOF. Assume $s = \max(p, q) \geqq 1$. Since $s \geqq p$ and p is the genus of $f(z)$, for a certain polynomial $P(z)$ of degree $\leqq s$ we get

$$f(z) = e^{P(z)} \prod_{n=1}^{\infty} E\left(\frac{z}{a_n}, s\right),$$

[3] Hereafter we assume that $h_1, h_2, \cdots$ are all some suitable fixed positive constants independent of r.

whence

$$\log^+ M(r, f) \leqq h_3 r^s + \log^+ \prod_{n=1}^{\infty} \left| E\left(\frac{z}{a_n}, s\right)\right|$$

$$\leqq h_3 r^s + h_4 \sum_{n=1}^{\infty} \frac{r^{s+1}}{|a_n|^s(|a_n| + r)} \tag{6}$$

$$\leqq h_5 \sum_{n=1}^{\infty} \frac{r^{s+1}}{|a_n|^s(|a_n| + r)}.$$

Similarly

$$\log^+ M(r, g) \leqq h_6 \sum_{n=1}^{\infty} \frac{r^{s+1}}{|b_n|^s(|b_n| + r)}. \tag{7}$$

Now, by the theory of meromorphic functions, we get

$$\frac{1}{r}\int_0^r \log^+ M(t)dt < C(k)T(kr, F), \tag{8}$$

where $T(r, F)$ is the characteristic function of $F(z)$, and $C(k)$ depends on a number $k>1$ only, and also we have

$$T(kr, F) = T(kr, f/g) \leqq T(kr, f) + T(kr, g) + O(1)$$
$$\leqq \log^+ M(kr, f) + \log^+ M(kr, g) + O(1). \tag{9}$$

Hence from (6), (7), (8), and (9) follows (4).

Next, assume $s=0$. Then by the same process we get (5).

PROOF OF THEOREM I. By the lemma, when $s \geqq 0$ we get

$$\frac{1}{r}\int_0^r \log^+ M(t)dt \leqq h_7 r^{s+1} \int_0^{\infty} \frac{N(t)}{t^{s+1}(t + r)}\, dt; \tag{10}$$

for, when $s \geqq 1$, from (4) this is obviously seen, and when $s=0$, from (5) we have

$$\frac{1}{r}\int_0^{\infty} \log^+ M(t)dt \leqq h_2 \int_0^{\infty} \frac{krN(t)}{t(t + kr)}\, dt$$

$$\leqq kh_2 \int_0^{\infty} \frac{rN(t)}{t(t + r)}\, dt.$$

On the other hand the integral function

$$G(z) = \prod_{n=1}^{\infty} E\left(\frac{z}{c_n}, s\right)$$

is of order $\rho=\max(\rho_1, \rho_2)$, since $c_1, c_2, \cdots$ are composed of $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$, whose exponents of convergence are ρ_1, ρ_2 respectively; and the genus of $G(z)$ is $s=\max(p, q)$, since $f(z)$ and $g(z)$ are of genera p and q respectively. Therefore by Shah[4]

$$\liminf_{r\to\infty} \frac{1}{N(r)\phi(r)}\, r^{s+1}\int_0^\infty \frac{N(t)}{t^{s+1}(t+r)}\,dt = 0$$

for $\rho=s\geqq 0$. Hence by (10) we get (2).

PROOF OF THEOREM II. The order and genus of integral function $G(z)$ are $\rho=\max(\rho_1, \rho_2)$ and $s=\max(p, q)$, respectively, as already seen, so that the exponent of convergence of $|c_1|, |c_2|, \cdots$ is ρ, and $\rho>0$, $s=[\rho]$, since ρ is not an integer.

Now set $\Phi(x)=h_1x^{-(s+1)}/1+x^{-1}$ for $1<\rho$, and $\Phi(x)=h_2\log(1+k/x)$ for $0<\rho<1$. Then $\Phi(x)$ is a positive and decreasing function for $x>0$, and we have

$$\Phi(x) < x^{-\rho+\eta} \qquad (0<\eta<\rho-s)$$

for all values of $x>0$ near $x=0$, and

$$\Phi(x) < x^{-\rho-\eta} \qquad (0<\eta<s+1-\rho)$$

for all values of x sufficiently great. Therefore by Pólya[5]

$$\liminf_{r\to\infty} \frac{1}{N(r)} \sum_{n=1}^{\infty} \Phi\left(\frac{|c_n|}{r}\right) \leqq \int_0^\infty \Phi(x^{1/\rho})\,dx < \infty.$$

Hence from the lemma

$$\liminf_{r\to\infty} \frac{1}{rN(r)} \int_0^r \log^+ M(t)\,dt \leqq \liminf_{r\to\infty} \frac{1}{N(r)} \sum_{n=1}^{\infty} \Phi\left(\frac{|c_n|}{r}\right) < \infty.$$

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[4] Shah, loc. cit. pp. 26–30.

[5] Pólya, loc. cit., Theorem VI, pp. 173–174.

REPRESENTATION OF FOURIER INTEGRALS AS SUMS. II

R. J. DUFFIN

Let the functions $f(x)$ and $g(x)$ be defined in terms of the function $\phi(x)$ by the sums:

$$f(x) = \sum_1^\infty \frac{\alpha_n}{n} \phi\left(\frac{x}{n}\right) \tag{1}$$

and

$$g(x) = \sum_1^\infty \frac{\alpha_n}{x} \phi\left(\frac{n}{x}\right). \tag{2}$$

Here $\alpha_n = \sin(\pi n/2)$, $n = 1, 2, \cdots$.

In a previous paper [1][1] it was shown that under quite weak restrictions on ϕ:

$$g(x) = \int_0^\infty \operatorname{sn} xt\, f(t)dt \tag{3}$$

$$f(x) = \int_0^\infty \operatorname{sn} xt\, g(t)dt. \tag{4}$$

Here sn x is an abbreviation for $\sin(\pi x/2)$.

This paper gives other conditions for the validity of these identities. The previous conditions permitted ϕ to have various types of discontinuity. The present paper is concerned with smooth functions; however, the growth at 0 and ∞ is permitted to be greater than before. Theorems 1, 2, and 3 of the previous paper together with Theorems 2 and 3 of this paper form a fairly complete elementary theory of these identities. The proofs given here do not depend on the previous paper.

The results of this paper hinge on the possibility of defining the Fourier sine transform for functions which do not vanish at infinity. Theorem 1 below shows that this is possible merely by employing summability. It is to be noted that Theorem 1 is not true, as it stands, for the cosine transform. For example, the cosine transform of any constant evaluated by such a definition would vanish. Hence the inversion formula could not apply. A theory of "generalized Fourier

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[1] Numbers in brackets refer to the references cited at the end of the paper.

integrals" has been developed by Fry, Hahn, Wiener, Burkill, Bochner, and others [2]. These writers do not differentiate between sine and cosine transforms, and it appears that their results contain no equivalent of Theorem 1.

THEOREM 1. *Let $f(x)$ be a function whose derivative $f'(x)$ is of bounded variation in the interval $(0, \infty)$, and let $f'(x)$ vanish at ∞. Then the integral (3) is summable C_1 to define a function $g(x)$. The integral (4) is improperly integrable at 0 and gives $f(x)$ for $x>0$.*

PROOF. A function which satisfies the conditions of the Theorem and also vanishes at 0 will be said to satisfy Conditions B. It is quite easy to see that the theorem is true for $f(x)$ a constant, so it is sufficient to prove the theorem under Conditions B. Two integrations by parts show that

$$\int_0^h \sin t\, f(t)dt = -\cos k\, f(k) + \sin k\, f'(k) - \int_0^h \sin t\, df'(t). \tag{5}$$

The integral on the right converges absolutely as $k\to\infty$. Therefore the integral on the left is summable C_1 provided the first two terms on the right are summable. But

$$\frac{1}{h}\int_0^h \cos k\, f(k)dk = \frac{\sin h}{h}\int_0^h f'(k)dk - \frac{1}{h}\int_0^h \sin k\, f'(k)dk.$$

Because f' vanishes at ∞ it follows that the Cesaro mean of the term $\cos k\, f(k)$ vanishes as $h\to\infty$. For the same reason the Cesaro mean of the second term on the right in (5) vanishes. Thus for $x>0$

$$\int_0^\infty \operatorname{sn} xt\, f(t)dt = -\frac{4}{\pi^2 x^2}\int_0^\infty \operatorname{sn} xt\, df'(t). \tag{6}$$

It is worth noting that a simpler type of summability will also establish (6). It is sufficient to take the limit in (5) for $k=\pi\nu/2$; $\nu=1, 3, 5, \cdots$. This causes the undesirable term $\cos k\, f(k)$ to drop out.

Denote the function defined by (6) as $g(x)$, and let b and c be positive constants such that $b\geqq 1\geqq c$. Then

$$\int_c^b \operatorname{sn} x\, g(x)dx = -\frac{4}{\pi^2}\int_0^\infty\int_c^b \frac{\operatorname{sn} x \operatorname{sn} xt}{x^2}\,dx\,df'(t). \tag{7}$$

The inner integral on the right may be written

$$\int_1^b \frac{\operatorname{sn} x \operatorname{sn} xt}{x^2}\,dx + \int_c^1 \frac{\operatorname{sn} x - \pi x/2}{x^2}\operatorname{sn} xt\,dx + \int_c^1 \frac{\pi \operatorname{sn} xt}{2x}\,dx.$$

It is obvious that the first two integrals converge uniformly in t as $b\to\infty$ and $c\to 0$. It is easy to show that the third integral converges uniformly for t in any finite range and that for all t it is uniformly bounded. Thus relation (7) is valid for $b=\infty$ and $c=0$ provided the integral on the left is evaluated as an improper integral at the lower limit. As is well known

$$\frac{4}{\pi^2}\int_0^\infty \frac{\operatorname{sn} x \operatorname{sn} xt}{x^2}\,dx = \begin{cases} t; & 0 \leqq t \leqq 1, \\ 1; & t \geqq 1. \end{cases} \tag{8}$$

Substituting (8) in (7) gives $f(1)=\int_0^\infty \operatorname{sn} x\, g(x)dx$. The observation that $f(kx)$ for $k>0$ satisfies the Conditions B and has the transform $g(x/k)/k$ clearly completes the proof.

THEOREM 2. *Let $\phi(x)$ be differentiable for $x\geqq 1$. Let $\phi'(x)$ be of bounded variation in the interval $(1, \infty)$, and let $\phi'(x)$ vanish at ∞. Let the same conditions be satisfied by $\Phi(x)=\phi(1/x)/x$. Then for $x>0$ the series* (1) *and* (2) *are summable C_1 and satisfy* (3) *and* (4). *The integrals are summable C_1 at ∞ and improperly integrable at* 0.

PROOF. Consider first the special case when $\phi(x)$ satisfies Conditions B. Let a function $f'(x)$ be defined by the series

$$f'(x) = \sum_1^\infty \frac{\alpha_n}{n^2}\phi'\left(\frac{x}{n}\right). \tag{9}$$

The function ϕ' is bounded, so the series is uniformly convergent and may be integrated termwise. Hence

$$\begin{aligned} \int_0^x f'(x)dx &= \sum_1^\infty \frac{\alpha_n}{n^2}\int_0^x \phi'\left(\frac{x}{n}\right)dx \\ &= \sum_1^\infty \frac{\alpha_n}{n}\,\phi\left(\frac{x}{n}\right). \end{aligned} \tag{10}$$

Thus the series (1) converges and defines a function $f(x)$ whose derivative $f'(x)$ is given by (9). From (10) it is seen that $f(x)$ vanishes at 0. From (9) it is seen that $f'(x)$ vanishes at ∞. Let V denote the operation of taking the variation on the interval $(0, \infty)$. Then from (9)

$$Vf'(x) \leqq \sum_1^\infty \frac{1}{n^2}\,V\phi'\left(\frac{x}{n}\right) \leqq 2V\phi'(x). \tag{11}$$

From these considerations it follows that $f(x)$ defined by (1) satisfies Conditions B if $\phi(x)$ does.

Let $f_m'(x)$ denote the first m terms of the series (9). Then

$$\int_0^\infty \operatorname{sn} t\, df_m'(t) = \sum_1^m \frac{\alpha_n}{n^2} \int_0^\infty \operatorname{sn} t\, d\phi'\left(\frac{t}{n}\right) = \int_0^\infty \sum_1^m \frac{\alpha_n}{n^2} \operatorname{sn} nt\, d\phi'(t). \tag{12}$$

From (11) it is clear that $V(f'-f_m')\to 0$ as $m\to\infty$; hence the integral on the left in (12) approaches $\int_0^\infty \operatorname{sn} t\, df'(t)$. The series on the right converges uniformly. Thus

$$\frac{4}{\pi^2}\int_0^\infty \operatorname{sn} t\, df'(t) = \int_0^\infty \sigma(t)\, d\phi'(t). \tag{13}$$

Here $\sigma(t) = \sum_1^\infty (4\alpha_n/(\pi n)^2) \operatorname{sn} nt$ is a well known series:

$\sigma(t) = t/2,\ -1 \leqq t \leqq 1;\quad \sigma(t) = (2-t)/2,\ 1 \leqq t \leqq 3;$ and so on.

Thus

$$-2\int_0^5 \sigma(t)\, d\phi'(t) + \phi'(5) = 2\int_0^5 \phi'(t)\, d\sigma(t)$$

$$= \int_0^1 \phi'(t)dt - \int_1^3 \phi'(t)dt + \int_3^5 \phi'(t)dt$$

$$= \phi(1) - [\phi(3) - \phi(1)] + [\phi(5) - \phi(3)].$$

But $\phi'(x)$ vanishes at ∞, so in general

$$-\int_0^\infty \sigma(t)\, d\phi'(t) = \lim \sum_1^\nu \alpha_n\phi(n) - \alpha_\nu\phi(\nu)/2;\quad \nu = 1, 3, 5, \cdots. \tag{14}$$

It is easy to show that a series summable by the method indicated in (14) is also summable by Cesaro means when use is made of the fact that $\phi(\nu)/\nu\to 0$.

Combining (6), (13), and (14) gives

$$\int_0^\infty \operatorname{sn} t\, f(t)dt = \sum_1^\infty \alpha_n\phi(n).$$

Replacing $\phi(x)$ by $\phi(x/k)/k$ in this relation proves (3). Theorem 1 guarantees (4).

Turning now to the general case, it is clearly possible to write $\phi(x) = \phi_1(x) + \phi_2(x)$ where $\phi_1(x)$ and $\Phi_2(x) = \phi_2(1/x)/x$ satisfy Conditions B. Thus

$$F_2(x) = \sum_1^\infty \frac{\alpha_n}{n}\,\Phi_2\left(\frac{x}{n}\right) = \sum_1^\infty \frac{\alpha_n}{x}\,\phi_2\left(\frac{n}{x}\right) = g_2(x).$$

Likewise $G_2(x)=f_2(x)$. Thus $f(x)=f_1(x)+G_2(x)$ and $g(x)=g_1(x)+F_2(x)$, so the validity of (3) is guaranteed by the validity of (3) and (4) in the special case. The same is true of (4).

THEOREM 3. *Let $\phi(z)$ be analytic at interior points of the sector $-\Delta\leqq\arg z\leqq\Delta$ for some $\Delta>0$, and in this sector suppose $|\phi(z)|=O(|z|^k+|z|^{-1-k})$, $0\leqq k<1$. Then:*

(a) *$\phi(x)$ satisfies the conditions of Theorem 2.*

(b) *The functions $f(z)$ and $g(z)$ satisfy the same conditions as $\phi(z)$ in the sector $-\delta\leqq\arg z\leqq\delta<\Delta$.*

(c) *If $k=0$ and $\phi(x)=o(1+|x|^{-1})$ at 0 and ∞, then $f(x)$ and $g(x)$ satisfy the same relation.*

(d) *In case* (c) *the series and integrals converge without recourse to summability.*

PROOF. Let z be a point in the sector δ; then by Cauchy's integral formula for $\phi''(z)$ with a circular contour of radius proportional to $|z|$, it follows that $|\phi''(z)|=O(|z|^{k-2}+|z|^{-2-k})$. Thus $\int_1^\infty|\phi''(x)|\,dx<\infty$, so $\phi'(x)$ is of bounded variation in the interval $(1,\infty)$. Similarly, Cauchy's formula for $\phi'(z)$ shows that $\phi'(x)$ vanishes at ∞. Clearly $\Phi(z)$ satisfies the same conditions as $\phi(z)$, so (a) is proved.

Formula (14) may be modified so as to hold for functions which do not vanish at 0. It is convenient to let $\rho(t)=1/2-\sigma(t)$. Then it is easy to see that for z in the sector δ

$$g(z)=\frac{1}{2z}\,\phi\left(\frac{1}{z}\right)+\frac{1}{z^2}\int_1^\infty\phi''\left(\frac{t}{z}\right)\rho(t)\,dt. \tag{15}$$

Introducing the bound for ϕ'' gives

$$|g(z)|\leqq\frac{1}{2|z|}\left|\phi\left(\frac{1}{z}\right)\right|+O\left(\frac{|z|^{-1-k}}{1-k}+\frac{|z|^k}{2+k}\right).$$

Thus $|g(z)|=O(|z|^k+|z|^{-1-k})$. The uniform convergence of (15) guarantees that $g(z)$ is analytic at interior points of the sector δ. Similar considerations apply to $f(z)$, so (b) is proved.

From (15) it is easy to see that if j is an integer of the form $4n+1$

$$g(z)=\sum_1^j\frac{\alpha_n}{z}\,\phi\left(\frac{n}{z}\right)-\frac{1}{2z}\,\phi\left(\frac{j}{z}\right)+\frac{1}{z^2}\int_j^\infty\phi''\left(\frac{t}{z}\right)\rho(t)\,dt. \tag{16}$$

But if $k=0$, $|\phi''(z)|=O(1+|z|^{-1})$, so

$$|g(x)|\leqq\sum_1^j\frac{1}{|x|}\left|\phi\left(\frac{n}{x}\right)\right|+O\left(\frac{|x|^{-1}}{j}+\frac{1}{2j^2}\right).$$

By choosing j sufficiently large, the last two terms may be made as small as desired for all $x \geqq 1$. By the assumption in (c) it is clear that the series vanishes as $x \to \infty$. Thus $g(x)$ vanishes at ∞. The same inequality shows that $x\,g(x)$ vanishes at 0. A similar argument applies to $f(x)$, so part (c) is proved.

It may be seen from (16) that since $\phi(x) \to 0$, the series for $g(x)$ converges in the ordinary sense. An integration by parts similar to that used in deriving (5) shows that because $f(k)$ and $g(k)$ vanish at ∞, the integrals converge in the ordinary sense at ∞. Part (b) shows that the integrals (4) and (5) are absolutely convergent at 0.

References

1. R. J. Duffin, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 447–455.

2. S. Bochner, *Vorlesungen über Fouriersche Integrale*, Leipzig, 1932, pp. 110 and 224.

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INEQUALITIES CONCERNING ULTRASPHERICAL POLYNOMIALS AND BESSEL FUNCTIONS

OTTO SZÁSZ[1]

1. **Introduction.** Ultraspherical polynomials are defined by the recurrence formula

$$(1.1)\quad (n+1)P_{n+1}^{\lambda}(x) = 2(n+\lambda)xP_n^{\lambda}(x) - (n+2\lambda-1)P_{n-1}^{\lambda}(x),$$

with $P_0^{\lambda}=1$, $P_1^{\lambda}=2\lambda x$. Thus we have

$$2P_2^{\lambda} = 2\lambda[2(1+\lambda)x^2 - 1],$$

and so on.

For $\lambda=1/2$ we get the Legendre polynomials $P_n(x)$. They satisfy the following interesting inequality:

$$(1.2)\quad \begin{aligned}\Delta_n(x) &\equiv [P_n(x)]^2 \\ &\quad - [P_{n-1}(x)][P_{n+1}(x)] \geqq 0, \qquad n \geqq 1,\ -1 \leqq x \leqq 1,\end{aligned}$$

with equality for $x=\pm 1$. This result is due to P. Turán, as stated in a recent paper by G. Szegö,[2] where four elegant proofs of the theorem are given. The first proof (similar to that of Turán) is based on Mehler's formula, while the third proof is based on the identity

$$\sum_{n=0}^{\infty} \frac{P_n(x)}{n!} z^n = e^{xz} J_0[(1-x^2)^{1/2}z],$$

and on the fact that the Bessel function J_0 has all its roots real.

Szegö remarks that the formula

$$\sum_{n=0}^{\infty} \frac{P_n^{\lambda}(x)}{P_n^{\lambda}(1)} \frac{z^n}{n!}$$
$$= 2^{\lambda-1/2}\Gamma(\lambda+1/2)e^{xz}[(1-x^2)^{1/2}z]^{1/2-\lambda}J_{\lambda-1/2}[(1-x^2)^{1/2}z],\ \lambda > -1/2,$$

and analogous formulas for Laguerre and Hermite polynomials yield inequalities analogous to (1.2) for ultraspherical, Laguerre, and

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[1] This paper was written at the Institute for Numerical Analysis of the National Bureau of Standards, with the financial support of the Office of Naval Research of the U. S. Navy Department.

[2] G. Szegö, *On an inequality of P. Turán concerning Legendre polynomials*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 401–405.

Hermite polynomials. Thus

$$\left[\frac{P_n^\lambda(x)}{P_n^\lambda(1)}\right]^2 - \frac{P_{n-1}^\lambda(x)P_{n+1}^\lambda(x)}{P_{n-1}^\lambda(1)P_{n+1}^\lambda(1)} \geqq 0, \qquad \lambda > -1/2. \tag{1.3}$$

We have,[3] for $\lambda > -1/2$,

$$P_n^\lambda(1) = C_{n+2\lambda-1,n} = \frac{2\lambda\Gamma(n+2\lambda)}{n!\Gamma(2\lambda+1)}, \quad n = 0, 1, \cdots, \tag{1.4}$$

so that (1.3) can be restated as

$$n(n+2\lambda)[P_n^\lambda(x)]^2 \geqq (n+1)(n+2\lambda-1)P_{n-1}^\lambda(x)P_{n+1}^\lambda(x);$$

with $\lambda = 1/2$, this reduces to the inequality (1.2).

We write $F_n(x)$ for $P_n^\lambda(x)/P_n^\lambda(1)$; then (1.3) becomes $\Delta_n^\lambda(x) \equiv [F_n(x)]^2 - F_{n-1}(x)F_{n+1}(x) \geqq 0$, and equality holds for $x = 1$. We have

$$F_0(x) \equiv 1, \quad F_1(x) = x, \quad F_2(x) = \frac{2(1+\lambda)x^2 - 1}{1+2\lambda}, \cdots.$$

For $0 < \lambda < 1$ we derive a sharper inequality, employing essentially the recursion formula (1.1). The result is

$$\frac{\lambda[1 - F_n^2(x)]}{(n+\lambda-1)(n+2\lambda)} \leqq \Delta_n^\lambda(x) < \frac{2\lambda\Gamma(2\lambda)}{\lambda+1}\,\frac{\Gamma(n)}{\Gamma(n+2\lambda)}, \tag{1.5}$$

$-1 \leqq x \leqq 1$. In particular, for $\lambda = 1/2$,

$$\frac{1 - [P_n(x)]^2}{(2n-1)(n+1)} \leqq \Delta_n(x) < \frac{2n+1}{3n(n+1)};$$

the smallness of Δ_n indicates that the inequality (1.2) is rather deep.

For Bessel functions we shall establish the inequality

$$J_\mu^2(t) - J_{\mu-1}(t)J_{\mu+1}(t) > \frac{1}{\mu+1}J_\mu^2(t), \qquad \mu > 0, \ t \text{ real}. \tag{1.6}$$

It seems particularly interesting that the procedure is very much the same in deriving (1.5) and (1.6), and is based mainly on the respective recurrence relations.

2. **Ultraspherical polynomials; $\lambda < 1$.** From (1.1) and (1.4), we obtain

[3] G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23, 1939, pp. 80, 166. Referred to as O. P.

$$(n+1)F_{n+1}(x) = 2(n+\lambda)xF_n(x)\frac{\Gamma(n+2\lambda)(n+1)}{\Gamma(n+1+2\lambda)} - (n+2\lambda-1)F_{n-1}(x)\frac{\Gamma(n+2\lambda-1)n(n+1)}{\Gamma(n+1+2\lambda)},$$

or

$$F_{n+1}(x) = \frac{2(n+\lambda)x}{n+2\lambda}F_n(x) - \frac{n}{n+2\lambda}F_{n-1}(x), \qquad n \geqq 1. \tag{2.1}$$

It follows that

$$F_{n+1}(x)F_n(x) = \frac{2(n+\lambda)}{n+2\lambda}xF_n^2(x) - \frac{n}{n+2\lambda}F_{n-1}(x)F_n(x). \tag{2.2}$$

From (2.1) we get

$$F_n(x) = \frac{2(n+\lambda-1)x}{n+2\lambda-1}F_{n-1}(x) - \frac{(n-1)}{n+2\lambda-1}F_{n-2}(x), \qquad n \geqq 2,$$

whence substitution into (2.2) yields

$$\begin{aligned}
&\frac{2x(n+\lambda)}{n+2\lambda}F_n^2(x)\\
&= F_{n+1}(x)\left\{\frac{2(n+\lambda-1)}{n+2\lambda-1}xF_{n-1}(x) - \frac{n-1}{n+2\lambda-1}F_{n-2}(x)\right\}\\
&\quad + \frac{n}{n+2\lambda}F_{n-1}(x)\left\{\frac{2(n+\lambda-1)}{n+2\lambda-1}xF_{n-1}(x) - \frac{n-1}{n+2\lambda-1}F_{n-2}(x)\right\}\\
&= \frac{2nx(n+\lambda-1)}{(n+2\lambda-1)(n+2\lambda)}F_{n-1}^2(x) + F_{n+1}(x)\frac{2(n+\lambda-1)}{n+2\lambda-1}xF_{n-1}(x)\\
&\quad - \frac{(n-1)}{n+2\lambda-1}F_{n-2}(x)\left\{\frac{2(n+\lambda)x}{n+2\lambda}F_n(x) - \frac{n}{n+2\lambda}F_{n-1}(x)\right\}\\
&\quad - \frac{(n-1)n}{(n+2\lambda-1)(n+2\lambda)}F_{n-2}(x)F_{n-1}(x)\\
&= \frac{2n(n+\lambda-1)x}{(n+2\lambda-1)(n+2\lambda)}F_{n-1}^2(x) + 2x\frac{n+\lambda-1}{n+2\lambda-1}F_{n-1}(x)F_{n+1}(x)\\
&\quad - \frac{2(n-1)(n+\lambda)x}{(n+2\lambda-1)(n+2\lambda)}F_{n-2}(x)F_n(x), \qquad n \geqq 2.
\end{aligned}$$

If we write

$$(2.3)\qquad \begin{aligned}&(n+\lambda)(n+2\lambda-1)F_n^2(x)\\&\quad-(n+\lambda-1)(n+2\lambda)F_{n-1}(x)F_{n+1}(x)=D_n^\lambda(x),\end{aligned}$$

then

$$\begin{aligned}D_n^\lambda(x)&=n(n+\lambda-1)F_{n-1}^2(x)-(n-1)(n+\lambda)F_{n-2}(x)F_n(x)\\&=\{(n+\lambda-1)(n+2\lambda-2)F_{n-1}^2(x)\\&\quad-(n+\lambda-2)(n+2\lambda-1)F_{n-2}(x)F_n(x)\}\\&\quad\cdot\frac{(n-1)(n+\lambda)}{(n+\lambda-2)(n+2\lambda-1)}+F_{n-1}^2(x)\\&\quad\cdot\left\{n(n+\lambda-1)-\frac{(n+\lambda-1)(n+2\lambda-2)(n-1)(n+\lambda)}{(n+\lambda-2)(n+2\lambda-1)}\right\}\\&=D_{n-1}^\lambda(x)\frac{(n-1)(n+\lambda)}{(n+\lambda-2)(n+2\lambda-1)}+(n+\lambda-1)F_{n-1}^2(x)\\&\quad\cdot\frac{n(n+\lambda-2)(n+2\lambda-1)-(n-1)(n+\lambda)(n+2\lambda-2)}{(n+\lambda-2)(n+2\lambda-1)}\\&=\frac{(n-1)(n+\lambda)}{(n+\lambda-2)(n+2\lambda-1)}D_{n-1}^\lambda(x)\\&\quad+\frac{2\lambda(\lambda-1)(n+\lambda-1)F_{n-1}^2(x)}{(n+\lambda-2)(n+2\lambda-1)}.\end{aligned}$$

Let

$$(2.4)\qquad g_n=g_{n-1}\frac{(n+\lambda-2)(n+2\lambda-1)}{(n-1)(n+\lambda)},\quad n\geqq 2,\ g_1=1;$$

then

$$(2.5)\qquad g_nD_n^\lambda(x)=g_{n-1}D_{n-1}^\lambda(x)+2\lambda(\lambda-1)\frac{n+\lambda-1}{(n-1)(n+\lambda)}g_{n-1}F_{n-1}^2(x),$$

where, from (2.3),

$$(2.6)\qquad \begin{aligned}D_1^\lambda(x)&=2\lambda(1+\lambda)F_1^2(x)-\lambda(1+2\lambda)F_0F_2(x)\\&=\lambda\{2(1+\lambda)x^2-2(1+\lambda)x^2+1\}\\&=\lambda.\end{aligned}$$

From (2.4), we have

$$g_n = \prod_2^n \frac{(\nu+\lambda-2)(\nu+2\lambda-1)}{(\nu-1)(\nu+\lambda)} = \frac{\lambda(\lambda+1)\prod_2^n(\nu+2\lambda-1)}{(n-1)!(n+\lambda-1)(n+\lambda)}$$

$$= \frac{\lambda(\lambda+1)}{(n+\lambda-1)(n+\lambda)} \prod_1^{n-1}\left(1+\frac{2\lambda}{\nu}\right) = O(n^{2\lambda-2}), \tag{2.7}$$

as $n \to \infty$.

It is well known [cf. O. P. pp. 80, 166] that, for $\lambda > 0$,

$$\max_{-1\leq x\leq 1} |P_n^\lambda(x)| = P_n^\lambda(1) = C_{n+2\lambda-1,n},$$

so that

$$|F_n(x)| \leqq F_n(1) = 1. \tag{2.8}$$

Hence

$$\frac{n+\lambda}{n(n+\lambda+1)} g_n F_n^2(x) \leqq \frac{g_n}{n} = O(n^{2\lambda-3}). \tag{2.9}$$

We now consider the case $0<\lambda<1$; it follows from (2.9) that the series

$$\sum_2^\infty \{g_{n-1}D_{n-1}^\lambda(x) - g_n D_n^\lambda(x)\}$$

is uniformly convergent in $-1 \leqq x \leqq 1$, and from (2.5), (2.6) we have

$$\lambda - \lim_{n\to\infty} g_n D_n^\lambda(x) = 2\lambda(1-\lambda)\sum_1^\infty \frac{(n+\lambda)g_n}{n(n+\lambda+1)} F_n^2(x).$$

We write

$$\lim_{n\to\infty} g_n D_n^\lambda(x) = \lambda\left\{1 - 2(1-\lambda)\sum_1^\infty \frac{(n+\lambda)g_n}{n(n+\lambda+1)} F_n^2(x)\right\} \tag{2.10}$$

$$\equiv \chi(x),$$

a continuous function of x in $-1 \leqq x \leqq 1$. It follows from (2.8) and (2.10) that

$$\chi(x) \geqq \chi(1) = \lambda\left\{1 - 2(1-\lambda)\sum_1^\infty \frac{(n+\lambda)g_n}{n(n+\lambda+1)}\right\}. \tag{2.11}$$

We next show that

$$\chi(1) = \lim g_n D_n^{\lambda}(1) = 0. \tag{2.12}$$

From (2.3), we have

$$D_n^{\lambda}(1) = (n+\lambda)(n+2\lambda-1) - (n+\lambda-1)(n+2\lambda) = \lambda, \tag{2.13}$$

and from (2.7), for $0<\lambda<1$, we see that $g_n \to 0$; this proves (2.12).

It now follows from (2.11) that

$$\chi(x) \geqq 0 = 1 - 2(1-\lambda)\sum_{1}^{\infty} \frac{(n+\lambda)g_n}{n(n+\lambda+1)},$$

so that

$$\sum_{1}^{\infty} \frac{(n+\lambda)g_n}{n(n+\lambda+1)} = \frac{1}{2(1-\lambda)}, \qquad 0<\lambda<1.$$

Note that by (2.7) we have

$$g_n = \frac{\lambda(\lambda+1)\Gamma(n+2\lambda)}{\Gamma(1+2\lambda)(n+\lambda-1)(n+\lambda)\Gamma(n)} \sim \frac{\lambda(\lambda+1)}{\Gamma(1+2\lambda)} n^{2\lambda-2}.$$

From (2.5) and (2.6) we get

$$g_n D_n^{\lambda}(x) = \lambda - 2\lambda(1-\lambda)\sum_{\nu=1}^{n-1} \frac{g_\nu(\nu+\lambda)F_\nu^2(x)}{\nu(\nu+\lambda+1)}; \tag{2.14}$$

furthermore

$$g_n D_n^{\lambda}(x) = \chi(x) + 2\lambda(1-\lambda)\sum_{n}^{\infty} \frac{g_\nu(\nu+\lambda)}{\nu(\nu+\lambda+1)} F_\nu^2(x).$$

Now from (2.8), (2.12), and (2.13) it follows that

$$\begin{aligned} g_n D_n^{\lambda}(x) &= 2\lambda(1-\lambda)\sum_{1}^{n-1} \frac{g_\nu(\nu+\lambda)}{\nu(\nu+\lambda+1)}\{1 - F_\nu^2(x)\} \\ &\quad + \sum_{n}^{\infty} \frac{g_\nu(\nu+\lambda)}{\nu(\nu+\lambda+1)} \geqq \sum_{n}^{\infty} \frac{g_\nu(\nu+\lambda)}{\nu(\nu+\lambda+1)} \\ &= g_n D_n^{\lambda}(1) = \lambda g_n. \end{aligned}$$

It follows from (2.14) that, uniformly in $-1 \leqq x \leqq 1$,

$$g_n D_n^{\lambda}(x) \downarrow \chi(x), \qquad \text{as } n \to \infty; \tag{2.15}$$

furthermore,

$$D_n^{\lambda}(x) \geqq D_n^{\lambda}(1) = \lambda. \tag{2.16}$$

From (2.15) we have

$$g_n D_n^{\lambda}(x) < g_1 D_1^{\lambda} = \lambda.$$

From (2.3), (2.13), and (2.16) we obtain

$$D_n^{\lambda}(x) = (n+\lambda-1)(n+2\lambda)\{F_n^2(x) - F_{n-1}(x)F_{n+1}(x)\} + \lambda F_n^2(x) \geqq \lambda,$$

so that

$$\Delta_n^{\lambda}(x) = F_n^2(x) - F_{n-1}(x)F_{n+1}(x) \geqq \frac{\lambda(1-F_n^2(x))}{(n+\lambda-1)(n+2\lambda)},$$

and

$$n^{2\lambda}\Delta_n^{\lambda}(x) \to \frac{\Gamma(1+2\lambda)}{\lambda(\lambda+1)}\chi(x),$$

uniformly in $-1 \leqq x \leqq 1$.

Summarizing, we have the following theorem:

THEOREM 1. *Denote by $P_n^{\lambda}(x)$ the ultraspherical polynomial and let*

$$F_n(x) = \frac{P_n^{\lambda}(x)}{P_n^{\lambda}(1)}, \qquad \Delta_n^{\lambda}(x) = F_n^2(x) - F_{n-1}(x)F_{n+1}(x).$$

If $0<\lambda<1$, then $n^{2\lambda}\Delta_n^{\lambda}(x)$ tends uniformly to a non-negative function, and

$$\frac{\lambda(1-F_n^2(x))}{(n+\lambda-1)(n+2\lambda)} \leqq \Delta_n^{\lambda}(x) < \frac{\Gamma(1+2\lambda)}{\lambda+1} \cdot \frac{\Gamma(n)}{\Gamma(n+2\lambda)}, \qquad -1 \leqq x \leqq 1.$$

In particular for $\lambda=1/2$ (the Legendre polynomial), we have

$$\frac{1-(P_n(x))^2}{(2n-1)(n+1)} \leqq (P_n(x))^2 - P_{n-1}(x)P_{n+1}(x) < \frac{2n+1}{3n(n+1)}. \tag{2.17}$$

3. **The case $\lambda>1$.** Let now $\lambda>1$; from (2.14) we have

$$g_n D_n^{\lambda}(x) = \lambda + 2\lambda(\lambda-1)\sum_{1}^{n-1} \frac{g_\nu(\nu+\lambda)}{\nu(\nu+\lambda+1)} F_\nu^2(x),$$

so that $g_n D_n^{\lambda}(x)$ increases as n increases. Hence

$$g_n D_n^{\lambda}(x) \geqq \lambda,$$

or

$$(n+\lambda-1)(n+2\lambda)g_n\Delta_n^{\lambda}(x) + \lambda g_n F_n^2(x) \geqq \lambda = D_n^{\lambda}(1).$$

Furthermore

$$g_n D_n^{\lambda}(x) \leqq g_n D_n^{\lambda}(1) = \lambda + 2\lambda(\lambda-1)\sum_{1}^{n-1} \frac{g_\nu(\nu+\lambda)}{\nu(\nu+\lambda+1)} = g_n\lambda.$$

Thus

$$\Delta_n^{\lambda}(x) \geqq \frac{\lambda}{(n+\lambda-1)(n+2\lambda)g_n} - \frac{\lambda F_n^2(x)}{(n+\lambda-1)(n+2\lambda)},$$

and

$$\Delta_n^{\lambda}(x) \leqq \frac{\lambda(1-F_n^2(x))}{(n+\lambda-1)(n+2\lambda)}.$$

In this case, from (2.7), we have $g_n \uparrow \infty$, and

$$\Delta_n^{\lambda}(x) = O(n^{-2}), \qquad \text{as } n \to \infty.$$

Finally, for $\lambda = 1$, we have

$$F_n(x) = \frac{\sin(n+1)\theta}{(n+1)\sin\theta}, \qquad x = \cos\theta,$$

$$g_n D_n^{\lambda}(x) = g_1 D_1^{\lambda} = 1, \qquad g_n = 1, \qquad D_n^{\lambda}(x) = 1,$$

$$n(n+2)\Delta_n(x) = 1 - F_n^2(x).$$

4. **An application.** If $P_{n+1}(x) = 0$, then from (2.17) we obtain

$$\frac{1-P_n^2(x)}{(n+1)(2n-1)} \leqq P_n^2(x) < \frac{2n+1}{3n(n+1)},$$

or

$$\frac{1}{n(2n+1)} \leqq P_n^2(x) < \frac{2n+1}{3n(n+1)}.$$

If $P_n(t) = 0$, then

$$\frac{1}{(n(2n+1))^{1/2}} \leqq |P_n(x)| = \left| \int_t^x P_n'(u)du \right|,$$

which yields a lower bound for $[x-t]$. To this end we employ the formula [O. P. p. 83]

$$\frac{d}{dx} P_n^{\lambda}(x) = 2\lambda P_{n-1}^{\lambda+1}(x),$$

so that, for $\lambda = 1/2$,

$$P_n'(x) = P_{n-1}^{3/2}(x);$$

hence, from (1.5), for $\lambda = 3/2$,

$$|P_n'(x)| \leqq C_{n+1,n-1} = (3/2)n(n+1);$$

equality holds for $x = \pm 1$.

It follows that

$$\frac{1}{(n(2n+1))^{1/2}} < |x-t|\,(3/2)n(n+1),$$

or

$$|x-t| > 2/3n^{-3/2}(2n+1)^{-1/2}(n+1)^{-1},$$

where $P_{n+1}(x) = 0$, $P_n(t) = 0$.

5. **Hermite polynomials.** The situation is simpler for Hermite polynomials $H_n(x)$, defined by

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad n = 1, 2, 3, \cdots,$$
$$H_0 = 1, \qquad H_1 = 2x.$$

In this case [see Amer. Math. Monthly vol. 55 (1948) solution of problem 4215, pp. 34–35],

$$\frac{1}{2^{n+1}n!}\{H_{n+1}^2 - H_nH_{n+2}\} = \sum_{\nu=0}^{n} \frac{1}{\nu!2^{\nu}} H_{\nu}^2,$$

hence monotone increasing, as n increases.

6. **Bessel functions.** The Bessel function of order μ is defined for $\mu > -1$ by the power series

$$J_{\mu}(t) = (t/2)^{\mu} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}(t/2)^{2\nu}}{\nu!\Gamma(\mu+\nu+1)}. \tag{6.1}$$

It satisfies the recursion formula

$$J_{\mu+1}(t) = \frac{2\mu}{t} J_\mu(t) - J_{\mu-1}(t), \qquad \mu > 0. \tag{6.2}$$

We write

$$\Lambda_\mu(t) = (2/t)^\mu \Gamma(\mu+1) J_\mu(t) = \Lambda_\mu(-t). \tag{6.3}$$

It then follows from (6.2) that

$$\frac{t^2 \Lambda_{\mu+1}(t)}{4\mu(\mu+1)} = \Lambda_\mu(t) - \Lambda_{\mu-1}(t), \tag{6.4}$$

and

$$\begin{aligned}\Lambda_\mu^2(t) &= \frac{t^2}{4\mu(\mu+1)} \Lambda_\mu(t)\Lambda_{\mu+1}(t) + \Lambda_{\mu-1}(t)\Lambda_\mu(t) \\ &= \frac{t^2}{4\mu(\mu+1)} \Lambda_{\mu+1}(t) \left\{ \frac{4\mu(\mu-1)}{t^2} (\Lambda_{\mu-1}(t) - \Lambda_{\mu-2}(t)) \right\} \\ &\quad + \Lambda_{\mu-1}(t)\Lambda_\mu(t).\end{aligned}$$

Thus, using (6.4) again, we have

$$\begin{aligned}\Lambda_\mu^2 = \frac{\mu-1}{\mu+1} &(\Lambda_{\mu+1}\Lambda_{\mu-1} - \Lambda_{\mu+1}\Lambda_{\mu-2}) \\ &+ \frac{4\mu(\mu-1)}{t^2} \Lambda_{\mu-1}(\Lambda_{\mu-1} - \Lambda_{\mu-2}),\end{aligned}$$

and

$$\begin{aligned}\Lambda_\mu^2 - \frac{\mu-1}{\mu+1} \Lambda_{\mu-1}\Lambda_{\mu+1} &= \frac{4\mu(\mu-1)}{t^2} \Lambda_{\mu-1}(\Lambda_{\mu-1} - \Lambda_{\mu-2}) \\ &\quad - \frac{\mu-1}{\mu+1} \Lambda_{\mu-2}\{\Lambda_\mu - \Lambda_{\mu-1}\} \frac{4\mu(\mu+1)}{t^2} \\ &= \frac{4\mu(\mu-1)}{t^2} \Lambda_{\mu-1}^2 - \frac{4\mu(\mu-1)}{t^2} \Lambda_{\mu-2}\Lambda_\mu.\end{aligned} \tag{6.5}$$

If we put

$$D_\mu(t) = \Lambda_\mu^2 - \frac{\mu-1}{\mu+1} \Lambda_{\mu-1}\Lambda_{\mu+1},$$

and

$$\Delta_\mu(t) = \Lambda_\mu^2 - \Lambda_{\mu-1}\Lambda_{\mu+1},$$

then we have

$$D_\mu = \frac{4\mu(\mu-1)}{t^2}\Delta_{\mu-1}, \tag{6.6}$$

and

$$\frac{\mu-1}{\mu+1}\Delta_\mu = D_\mu - \frac{2}{\mu+1}\Lambda_\mu^2.$$

Now from (6.5) we get

$$D_\mu(t) = \frac{4\mu^2(\mu-1)}{(\mu-2)t^2}D_{\mu-1}(t) - \frac{8\mu(\mu-1)}{(\mu-2)t^2}\Lambda_{\mu-1}^2 \qquad (\mu \neq 2);$$

hence

$$\frac{t^{2\mu}D_\mu(t)}{4^\mu(\mu-1)\Gamma^2(\mu+1)} = \frac{t^{2(\mu-1)}D_{\mu-1}(t)}{4^{\mu-1}(\mu-2)\Gamma^2(\mu)} - \frac{2t^{2(\mu-1)}\Lambda_{\mu-1}^2}{4^{\mu-1}\mu(\mu-2)\Gamma^2(\mu)}. \tag{6.7}$$

It follows that the sequence

$$Q_{\mu+\nu}(t) = \frac{t^2(\mu+\nu)D_{\mu+\nu}(t)}{4^{\mu+\nu}(\mu+\nu-1)\Gamma^2(\mu+\nu+1)}, \qquad \nu = 0, 1, 2, \cdots,$$

is monotone decreasing as ν increases. Furthermore

$$D_1(t) = \Lambda_1^2(t), \qquad D_2(t) = \frac{8}{t^2}\Delta_2(t).$$

It is known[4] that

$$|\Lambda_\mu(t)| \leqq \Lambda_\mu(0) = 1, \qquad t \text{ real};$$

hence

$$Q_\mu(t) \to 0 \qquad \text{as } \mu \to \infty,$$

and, from (6.7),

$$Q_\mu(t) = 2\sum_{\nu=0}^{\infty} \frac{t^{2(\mu+\nu)}\Lambda_{\mu+\nu}^2(t)}{4^{\mu+\nu}(\mu+\nu-1)\Gamma^2(\mu+\nu+1)}, \qquad \mu > 1,$$

[4] See S. Minakshisundaram and Otto Szász, *On absolute convergence of multiple Fourier series*, Trans. Amer. Math. Soc. vol. 61 (1947) pp. 36–53, in particular formula (2.3).

the series being uniformly convergent. It now follows that

$$D_\mu(t) > 0, \qquad \text{for } \mu > 1,\ t \text{ real};$$

hence, from (6.6), we have

$$\Delta_{\mu-1}(t) > 0, \qquad \mu > 1,$$

or, in terms of the Bessel functions,

$$J_\mu^2(t) - J_{\mu-1}(t)J_{\mu+1}(t) > \frac{1}{\mu+1} J_\mu^2(t), \qquad \mu > 0,\ t \text{ real}.$$

If, for example, $\mu = 3/2$, we get

$$t \sin^2 t - 3 \cos t(\sin t - t \cos t) > 0 \qquad \text{for } t > 0.$$

We state the main result of this section as:

THEOREM 2. *If $J_\mu(t)$ denotes the Bessel function of index μ, defined by* (6.1), *then, for $\mu > 0$ and all real values of t,*

$$J_\mu^2(t) - J_{\mu-1}(t)J_{\mu+1}(t) > \frac{1}{\mu+1} J_\mu^2(t).$$

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ON BOUNDED ANALYTIC FUNCTIONS

ZEEV NEHARI

The objective of this paper is to give an alternative derivation of results on bounded analytic functions recently obtained by Ahlfors [1] and Garabedian [2].[1] While it is admitted that the main idea to be used is more in the nature of a lucky guess than of a method, it will be found that the gain in brevity and simplicity of the argument is considerable. As a by-product, we shall also obtain a number of hitherto unknown identities between various domain functions.

The basic problem treated in the above-mentioned papers is the following generalization of the classical Schwarz lemma: Given a finite schlicht domain D of connectivity n ($n \geqq 1$) in the complex z-plane and a point ζ in D, to find a function $F(z)$ with the following properties: (a) $F(z)$ belongs to the family B of analytic functions $f(z)$ which are single-valued and regular in D and satisfy there $|f(z)| \leqq 1$; (b) $|F'(\zeta)| \geqq |f'(\zeta)|$, where $f(z)$ is any function in B. Evidently, it is sufficient to solve this problem for any domain D' which is conformally equivalent to D. In particular, we may therefore assume, without restricting the generality of what follows, that D is bounded by analytic curves.

It was shown by Ahlfors that $F(z)$ yields a $(1, n)$ conformal mapping of D onto the interior of the unit circle and that $n-1$ of the n zeros of $F(z)$ coincide with the zeros of a single-valued function $h(z)$ which is regular in D with the exception of a simple pole at $z=\zeta$ and satisfies $-ih(z)dz>0$ on the boundary Γ of D; the nth zero of $F(z)$ is located at $z=\zeta$. It was subsequently noticed by Garabedian that the function $h(z)$ can be written in the form $h(z)=F(z)q(z)$ where $q(z)-(z-\zeta)^{-2}$ is regular in D and that the extremal property of $F(z)$ can be deduced in a very elegant manner from the resulting inequality

$$\frac{1}{i}F(z)q(z)dz > 0, \qquad z \in \Gamma. \tag{1}$$

Indeed, if $f(z)$ is in B and is continuous on Γ, we have, by the residue theorem,

$$|f'(\zeta)| = \left|\frac{1}{2\pi i}\int_\Gamma f(z)q(z)dz\right| \leqq \frac{1}{2\pi}\int_\Gamma |q(z)dz| = \frac{1}{2\pi}\int_\Gamma |F(z)q(z)dz|$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

$$= \frac{1}{2\pi i}\int_\Gamma F(z)q(z)dz = F'(\zeta),$$

where (1) and the fact that, on Γ, $|F(z)| \equiv 1$ have been used.

The inequality (1) not only solves the problem stated above but, as likewise shown by Garabedian, it may also be used in order to establish the connection between $F(z)$ and the Szegö kernel function [4] of D. This connection, in turn, leads to a possibility of effectively computing $F'(\zeta)$ in terms of a certain complete orthonormal system of functions in D.

Everything depends therefore on the proof of the existence of two functions $F(z)$ and $q(z)$, where $q(z)$ has the required double pole and $|F(z)| \equiv 1$ on Γ, such that (1) is satisfied for $z \in \Gamma$. We shall carry through this proof by giving, with the help of an allied simple extremal problem, a construction of these functions in terms of the Green's function and the harmonic measures of D.

We use the following notations: $p(z) = p(z, \zeta) = g(z, \zeta) + ih(z, \zeta)$ denotes the analytic function whose real part is the Green's function $g(z, \zeta)$ of D, that is, the harmonic function which is regular in D, apart from the point $z = \zeta$ where $g(z, \zeta) + \log|z - \zeta|$ is regular, and vanishes for $z \in D$. $w_\nu(z) = \omega_\nu(z) + i\omega_\nu^*(z)$ denotes the analytic function whose real part is the harmonic measure $\omega_\nu(z)$ of the boundary component Γ_ν $(\nu = 1, 2, \cdots, n)$, that is, that harmonic function which is regular in D, is equal to 1 for $z \in \Gamma_\nu$ and vanishes on all the other boundary continua. We further use the notations $u(z) = u(z, \zeta) = \partial p(z, \zeta)/\partial\xi$, $v(z) = v(z, \zeta) = (1/i)\partial p(z, \zeta)/\partial\eta$, $\zeta = \xi + i\eta$. It is easily confirmed that, with the exception of a simple pole of residue 1 at $z = \zeta$, both $u(z)$ and $v(z)$ are free of singularities in D and that, furthermore, $\mathrm{Re}\{u(z, \zeta)\} \equiv \mathrm{Im}\{v(z, \zeta)\} \equiv 0$ for $z \in \Gamma$.

Since the real parts of the functions $p(z)$, $w_\nu(z)$, $u(z)$, $iv(z)$ are constant on Γ, the differentials

$$\frac{1}{i}p'(z)dz, \quad \frac{1}{i}w_\nu'(z)dz, \quad \frac{1}{i}u'(z)dz, \quad v'(z)dz$$

are real there. The same is therefore also true of a linear combination

$$\frac{1}{i}t(z)dz$$

$$= \frac{1}{i}\left[\cos 2\lambda u'(z) + i\sin 2\lambda v'(z) - \alpha p'(z) + \sum_{\nu=1}^{n-1}\mu_\nu w_\nu'(z)\right]dz \tag{2}$$

with real λ, α, $\mu_1, \cdots, \mu_{n-1}$. Since Γ consists of analytic curves, the

functions $u(z), v(z), p(z), w_\nu(z)$ are regular on Γ—an immediate consequence of the Schwarz reflection principle. It is further well known, and easily proved, that $ip'(z)dz$ is not only real but also positive on Γ. If, in (2), $\lambda, \mu_1, \cdots, \mu_{n-1}$ are kept constant and α ($\alpha > 0$) is taken large enough, the differential $t(z)dz$ will therefore satisfy

$$\frac{1}{i} t(z)dz \geqq 0, \qquad z \in \Gamma. \tag{3}$$

We now pose the following extremal problem: For given λ, to find the differential (2) which minimizes α under the condition (3). It was shown by the author [3] that this problem has a solution, say $i^{-1}t_\lambda(z)dz$, and that the function $t_\lambda(z)$ belonging to this extremal differential has a double zero on each boundary component Γ_ν, $\nu = 1, 2, \cdots, n$.

It is readily confirmed that, in the vicinity of $z=\zeta$, the function $t_\lambda(z)$ has the expansion

$$t_\lambda(z) = -\frac{e^{2\Omega}}{(z-\zeta)^2} - \frac{\alpha}{z-\zeta} + \text{regular terms.} \tag{4}$$

We shall now show that $t_\lambda(z)$ is free of zeros in D. In view of (3), the expression $t_\lambda(z)dz$ does not change its argument if z describes the boundary Γ of D. If $\Delta \arg t_\lambda$ denotes the total change of the argument of t_λ along Γ, and $\Delta \arg dz$ has a similar meaning, we have therefore

$$\Delta \arg t_\lambda = -\Delta \arg dz.$$

Since $\Delta \arg dz = -2\pi(n-2)$, it follows that

$$\frac{1}{2\pi} \Delta \arg t_\lambda = n - 2. \tag{5}$$

By the argument principle, this is equal to the number of zeros of t_λ in D minus the number of its poles there, plus a contribution of $1/2$ for each zero of t_λ on Γ. Since t_λ has n double zeros on Γ and a double pole at $z=\zeta$, it follows from (5) that t_λ has no zeros in D and, moreover, that t_λ does not vanish on Γ at points different from the n double zeros already located.

Our next step is to prove that the function

$$\sigma_\lambda(z) = (t_\lambda(z))^{1/2} = \frac{ie^{\Omega}}{z-\zeta} + \text{regular terms} \tag{6}$$

is single-valued in D. For this purpose we have to show that, in addition to the fact just established that $\sigma_\lambda(z)$ has no zeros in D, the argu-

ment of the function $\sigma_\lambda(z)$ returns to its initial value if z describes a boundary component Γ_ν $(\nu=1, \cdots, n)$ of D. This is equivalent to showing that the increment of $\arg t_\lambda$ along Γ_ν is an even multiple of 2π. That this is indeed the case is easily shown as follows: If we start from the point on Γ_ν at which the double zero of t_λ is located and describe Γ_ν in the positive sense with respect to D, the total variation of $\arg t_\lambda$ is, in view of (3), equal to the negative value of the total variation of $\arg dz$, that is, it is either 2π or -2π. At the double zero of t_λ, $\arg t_\lambda$ jumps by 2π. The variation of t_λ along the whole of Γ_ν is therefore either 0 or 4π. Hence, the function $\sigma_\lambda(z)$ defined in (6) is indeed regular and single-valued in D.

We now introduce two functions $K_\lambda(z, \zeta)$ and $L_\lambda(z, \zeta)$ by the definitions

$$
\begin{aligned}
K(z, \zeta) = K_\lambda(z, \zeta) &= \frac{ie^{\Omega}}{4\pi}(\sigma_\lambda + i\sigma_{\lambda+\pi/2}) \\
L(z, \zeta) = L_\lambda(z, \zeta) &= \frac{e^{-\Omega}}{4\pi i}(\sigma_\lambda - i\sigma_{\lambda+\pi/2}).
\end{aligned} \tag{7}
$$

Both functions have also been written without the subscript λ, in anticipation of the fact—to be proved further below—that they are independent of the parameter λ. It is easily confirmed that $K(z, \zeta)$ is regular in D and that $L(z, \zeta)$ has a simple pole at $z=\zeta$ with the residue $(2\pi)^{-1}$.

For $z\in\Gamma$, the two functions $K(z)=K(z, \zeta)$ and $L(z)=L(z, \zeta)$ are connected by the relation

$$
\frac{1}{i}L(z)dz = K^*(z)ds, \tag{8}
$$

where $ds=|dz|$ is the length element on Γ.[2] We shall prove (8) by showing that both the arguments and the absolute values on both sides are equal. We have, by (6), (7) and (3),

$$
\frac{1}{i}K(z)L(z)dz = \frac{1}{16\pi^2}(\sigma_\lambda^2 + \sigma_{\lambda+\pi/2}^2)dz \geqq 0, \tag{9}
$$

whence

$$
\arg\frac{1}{i}L(z)dz = -\arg K(z) = \arg K^*(z) = \arg K^*(z)ds,
$$

which shows that the arguments are equal. Again by (7), we have

[2] The asterisk denotes conjugate complex quantities.

$$\frac{K(z)}{L(z)} = -e^{2\Omega}\left(\frac{\sigma_\lambda}{\sigma_{\lambda+\pi/2}} + i\right)\left(\frac{\sigma_\lambda}{\sigma_{\lambda+\pi/2}} - i\right)^{-1}. \tag{10}$$

In view of (3) and (6), we have

$$\left(\frac{\sigma_\lambda}{\sigma_{\lambda+\pi/2}}\right)^2 = \frac{(1/i)t_\lambda dz}{(1/i)t_{\lambda+\pi/2}dz} \geqq 0. \tag{11}$$

As a result, $\sigma_\lambda\sigma_\lambda^{-1}{}_{+\pi/2}$ is real on Γ, whence, by (10), $|KL^{-1}| \equiv 1$ for $z \in \Gamma$. This completes the proof of (8).

As a first application of (8), we shall show that $K(z)$ and $L(z)$ are independent of λ and that we were therefore justified in dropping the subscript λ in (7). For suppose there are two different pairs of functions K_1, L_1 and K_2, L_2 belonging to different values of λ. Obviously, the function $L_1 - L_2$ is regular in $D+\Gamma$, whence, by Cauchy's theorem,

$$\int_\Gamma [L_1(z) - L_2(z)][K_1(z) - K_2(z)]dz = 0.$$

In view of (8), this is equivalent to

$$\int_\Gamma |K_1(z) - K_2(z)|^2 ds = 0,$$

whence $K_1(z) \equiv K_2(z)$ on Γ and therefore throughout D. The identity $L_1(z) \equiv L_2(z)$ follows by another application of (8).

The fact that both $K(z)$ and $L(z)$ are independent of λ leads to an interesting identity. With the notations

$$\sigma(z) = \sigma_0(z), \qquad \tau(z) = \sigma_{\pi/2}(z) \tag{12}$$

we have, by (7),

$$e^{\Omega}(\sigma_\lambda + i\sigma_{\lambda+\pi/2}) = \sigma + i\tau$$
$$e^{-\Omega}(\sigma_\lambda - i\sigma_{\lambda+\pi/2}) = \sigma - i\tau$$

whence

$$\sigma_\lambda(z) = \cos\lambda\sigma(z) + \sin\lambda\tau(z). \tag{13}$$

Consider now the function

$$T(z) = \sigma(z)/\tau(z).$$

By (11) and (12), $T(z)$ is real on Γ. At interior points of D, $T(z)$ cannot be real. Indeed, suppose $T(z) = -\tan\lambda_0$, where λ_0 is an arbitrary real number between $-\pi/2$ and $\pi/2$. This would entail $\cos\lambda_0\sigma(z)$

$+\sin \lambda_0 \tau(z) = 0$. By (13), this would mean that $\sigma_{\lambda_0}(z)$ has a zero in D, which further above was shown not to be true. The same argument also shows that every real value is taken by $T(z)$ on Γ exactly n times, this being the number of zeros of $\sigma_\lambda(z)$ on Γ. Since $T(z)$ is real on Γ but at no interior point of D, it follows therefore that $w = T(z)$ maps D onto an n times covered half-plane which is bounded by the real axis. If, in (6), the positive value of the square root is taken, we have, by (6) and (12), $T(\zeta) = -i$, which shows that $w = T(z)$ yields a $(1, n)$ mapping of D onto the lower half-plane, $\operatorname{Im}\{w\} < 0$. Consequently, the function

$$F(z) = -\frac{T(z) + i}{T(z) - i} = -\frac{\sigma(z) + i\tau(z)}{\sigma(z) - i\tau(z)} = \frac{K(z)}{L(z)} \tag{14}$$

yields a $(1, n)$ mapping of D onto the unit circle. This, together with (9) and the argument principle, incidentally shows that $K(z)$ has $n-1$ zeros in D and that $L(z)$ is free of zeros there.

If we introduce the function $q(z)$ by

$$q(z) = 4\pi^2 L^2(z) \tag{15}$$

we have, by (14) and (9),

$$\frac{1}{i} F(z)q(z)dz = \frac{4\pi^2}{i} K(z)L(z)dz \geqq 0, \qquad z \in \Gamma. \tag{16}$$

Comparison of (16) with (1) shows that $F(z)$ and $q(z)$ are identical with the functions $F(z)$ and $q(z)$ introduced there, provided we can show that the principal part of $q(z)$ at $z = \zeta$ is $(z-\zeta)^{-2}$. This is indeed the case. For suppose the principal part of $q(z)$ at this point is $(z-\zeta)^{-2} + \gamma(z-\zeta)^{-1}$. By the residue theorem and in view of $|F(z)| \equiv 1$ $(z \in \Gamma)$, we have

$$\gamma = \frac{1}{2\pi i} \int_\Gamma q(z)dz = \frac{1}{2\pi i} \int_\Gamma F^*(z)F(z)q(z)dz.$$

By virtue of (16), this may also be written

$$\gamma^* = \frac{1}{2\pi i} \int_\Gamma F^2(z)q(z)dz.$$

Since $F(z)$ has a zero at $z = \zeta$—originating from the simple pole of $L(z)$—the integral at the right-hand side vanishes by the Cauchy theorem, whence $\gamma = 0$. This completes the proof of the identity of the functions $F(z)$ and $q(z)$ with the functions—denoted by the same symbols—introduced at the beginning of this paper.

It was shown by Garabedian that the function $(2\pi)^{-1}F(z)[q(z)]^{1/2}$ is identical with the Szegö kernel function [4] of D. In view of (14) and (15), this function is identical with $K(z)$. The identity of $K(z)$ $=K(z, \zeta)$ with the Szegö kernel function is also easily shown with the help of (8). Indeed, if $f(z)$ is a function which is regular and single-valued in D and continuous in $D+\Gamma$, we have, by (8) and the Cauchy theorem,

$$f(\zeta) = \frac{1}{i}\int_\Gamma f(z)L(z)dz = \int_\Gamma K^*(z, \zeta)f(z)ds, \tag{17}$$

and this relation is characteristic of the Szegö kernel function.

The Szegö kernel function $K(z, \zeta)$ can be effectively computed in terms of a certain complete orthonormal set of functions [4]. If $K(z, \zeta)$ is known, $L(z)=L(z, \zeta)$ can also be immediately determined. We have, by (8) and the residue theorem,

$$\int_\Gamma \frac{K^*(t, \zeta)ds_t}{t-z} = \frac{1}{i}\int_\Gamma \frac{L(t, \zeta)dt}{t-z} = \frac{1}{\zeta - z} + 2\pi L(z, \zeta),$$

whence

$$2\pi L(z, \zeta) = \frac{1}{z-\zeta} + \int_\Gamma \frac{K^*(t, \zeta)ds_t}{t-z}. \tag{18}$$

Once the kernel function is known, we are thus in a position to determine explicitly the functions $F(z)$ and $q(z)$, as well as the functions $\sigma_\lambda(z)$ mentioned before.

If we combine (2), (6), (7), (13), (14), (15), (18), a large number of interesting identities connecting $K(z)$, $L(z)$, $F(z)$, $q(z)$, $\sigma_\lambda(z)$ with the Green's function and the harmonic measures can be obtained. As an example, we prove the identity

$$\frac{2\pi K(z, \zeta)L(z, \zeta)}{K(\zeta, \zeta)} = p'(z, \zeta) + \sum_{\nu=1}^{n-1} c_\nu w_\nu'(z), \tag{19}$$

where, as in (2), Re $\{p(z, \zeta)\}$ is the Green's function of D, Re $\{w_\nu(z)\}$ the harmonic measures of Γ_ν $(\nu=1, \cdots, n-1)$ with respect to D, and the c_ν are suitable real constants. Indeed, by (9), (6), (2), and (12)

$$\frac{2\pi K(z, \zeta)L(z, \zeta)}{K(\zeta, \zeta)} = \frac{\sigma^2(z) + \tau^2(z)}{8\pi K(\zeta, \zeta)} = Cp'(z, \zeta) + \sum_{\nu=1}^{n-1} c_\nu w_\nu'(z),$$

where $C, c_1, c_2, \cdots, c_{n-1}$ are constants. Since the residue of the simple

pole of $L(z, \zeta)$ at $z=\zeta$ is $(2\pi)^{-1}$, the value of constant C is immediately seen to be 1. This completes the proof of the identity (19).

The constants c_ν in (19) can be expressed in terms of the harmonic measures $\omega_\nu(z)$, their periods $P_{\mu\nu}$ with respect to the boundary components Γ_μ, and the kernel function. By (8), we have

$$\int_{\Gamma_\mu} K(z, \zeta)L(z, \zeta)dz = \int_{\Gamma_\mu} |K(z, \zeta)|^2 ds.$$

On the other hand,

$$\int_{\Gamma_\mu} p'(z, \zeta)dz = 2\pi i\omega_\nu(\zeta)$$

and

$$\int_{\Gamma_\mu} w_\nu'(z)dz = 2\pi i P_{\mu\nu}.$$

In view of (19), the constants c_ν are therefore the solutions of the system of linear equations

$$[K(\zeta, \zeta)]^{-1} \int_{\Gamma_\mu} |K(z, \zeta)|^2 ds = \omega_\mu(\zeta) + \sum_{\nu=1}^{n-1} P_{\mu\nu}c_\nu, \qquad \mu = 1, \cdots, n-1.$$

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NONLINEAR DIFFERENCE EQUATIONS CONTAINING A PARAMETER

C. F. STEPHENS

1. **Formulation of the problem.** Consider the system of nonlinear difference equations having the form

$$y_i(x+1) = x^{-1}\sum_{j=1}^{n} b_{ij}(x)y_j(x) + f_i(y_1(x), \cdots, y_n(x); p; x), \tag{1}$$

$$f_i(0, \cdots, 0; 0; x) \equiv 0 \qquad (i = 1, \cdots, n),$$

where the f_i are analytic functions of $y_i(x)$ and the constant parameter p, continuous functions of the complex variable x, and bounded for all values of the $y_i(x)$, p, and x for which

$$G: \qquad |y_i(x)| \leqq r_i; \qquad |p| \leqq \rho; \qquad |x| \geqq K.$$

The f_i contain terms in p alone and can be developed into convergent power series in the $y_i(x)$ and p. The $b_{ij}(x)$ are any continuous and bounded functions of x which are defined in the domain G. We could have taken the parameter p to be a periodic function of period 1.

In other words, we may write

$$f_i = g_i(x, p) + h_i(y_1(x), \cdots, y_n(x); p; x), \tag{2}$$

where

$$g_i(x, p) = x^{-1}A_{i1}(x)p + \sum_{j=2}^{\infty} A_{ij}(x)p^j, \tag{2a}$$

$$h_i = \sum_{m=2}^{\infty} \Sigma a^{(i)}_{\alpha_0, \alpha_1, \cdots, \alpha_n}(x)p^{\alpha_0}y_1^{\alpha_1}(x) \cdots y_n^{\alpha_n}(x), \tag{2b}$$

and

$$\Sigma = (\alpha_0 \neq m; \alpha_0, \cdots, \alpha_n \geqq 0; \alpha_0 + \cdots + \alpha_n = m).$$

The $A_{ij}(x)$ $(j=1, 2, \cdots)$ and $a^{(i)}_{\alpha_0, \alpha_1, \cdots, \alpha_n}(x)$ are any continuous and bounded functions of x such that the right members of (2a) and (2b) converge, where the functions $A_{i1}(x)$ can conceivably be identically zero.

In a recent work[1] the author made certain investigations concern-

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[1] C. F. Stephens, *Nonlinear difference equations analytic in a parameter*, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 268–282.

ing systems of nonlinear difference equations similar to system (1). However, the functions in the right members of these systems corresponding to the f_i are more restrictive as functions of x. A system similar to (1) in which the functions corresponding to the $b_{ij}(x)$ are more restrictive as functions of x was first considered by the author in his doctoral dissertation, which was prepared at the University of Michigan under the supervision of J. A. Nyswander. Also, the results contained in the first two chapters of the author's paper to which reference is made, with slight changes, were obtained in his doctoral thesis, Theorem 6 being a generalization of a theorem considered there. The procedure for investigating solutions of system (1) which will be given in this paper is different from and simpler than the one considered in the author's dissertation.

2. **Preliminary results.** The proof of the existence of a solution of system (1) which is analytic in p and continuous in x is based on the following result. Let us consider the system of nonlinear difference equations

$$
\begin{aligned}
&z_i(x+1) = l_i(z_1(x), \cdots, z_n(x); p; x), \\
&l_i(0, \cdots, 0; p; x) = a_i(x, p) \not\equiv 0 \qquad (i = 1, \cdots, n),
\end{aligned} \tag{3}
$$

where the l_i are analytic functions of the $z_i(x)$, continuous functions of x and p, and can be developed into convergent power series starting with terms independent of $z_i(x)$.

Let C be a circle with center at the origin and radius $R'(=R+1)$. Draw tangents t and t' to the circle C at the points where the circle cuts the axis of imaginaries. Denote by D all points not interior to the domain bounded by that part of the circle C which lies to the left of the axis of imaginaries and the parts of tangents t and t' which do not lie to the left of this axis. Thus the domain D contains no part of the positive axis of reals.

The following lemma is similar to a result obtained previously by the author,[2] and can be proved by similar methods:

LEMMA. *If we can determine positive constants* λ, R, *and* r *such that*

$$
|\, l_i(z_1(x), \cdots, z_n(x); p; x)\,| < r/4n
$$

for

$$
|\, z_i(x)\,| \leqq r, \qquad |\, p\,| \leqq \lambda, \qquad |\, x\,| \geqq R,
$$

then the system of equations (3) *has a solution having the form*

[2] Ibid. §9.

$$z_i(x) = \sum_{k=1}^{\infty} C_i^{(k)}(x, p) \qquad (i = 1, \cdots, n) \tag{4}$$

where the right members of equations (4) *are absolutely and uniformly convergent in* x *and* p *for* x *in the domain* D *and for* $|p| \leqq \lambda$.

For the proof, a new system of equations

$$z_i(x+1) = tl_i(z_1(x), \cdots, z_n(x); p; x) \qquad (i = 1, \cdots, n) \tag{5}$$

is introduced, containg a new parameter t. A solution

$$z_i(x) = \sum_{k=1}^{\infty} C_i^{(k)}(x, p)t^k$$

of this system is computed by substitution into (5) and comparison of like powers of t. By a method of dominant analytic implicit equations the series is shown to converge for $t=1$, and when t is set equal to 1 a solution of (3) is obtained in the form (4).

3. **Proof.** We shall now prove the existence of a solution of system (1) which is analytic in the parameter p and continuous in x. In order to accomplish this aim, we shall transform the system of equations (1) into a system such that the conditions of our lemma are satisfied. Let us make the transformation

$$y_i(x) = pz_i(x) \qquad (i = 1, \cdots, n). \tag{6}$$

Then the system of equations (1) is transformed into the system

$$\begin{aligned} z_i(x+1) &= x^{-1}\sum_{j=1}^{n} b_{ij}(x)z_j(x) + x^{-1}A_{i1}(x) + p\sum_{j=2}^{\infty} A_{ij}(x)p^{j-2} \\ &\quad + p\sum_{m=2}^{\infty}\Sigma a^{(i)}_{a_0,\cdots,a_n}(x)p^{m-2}z_1^{a_1}(x)\cdots z_n^{a_n}(x) \\ &\equiv l_i(z_1(x), \cdots, z_n(x); p; x) \qquad (i = 1, \cdots, n). \end{aligned} \tag{7}$$

Let r be the smallest of the numbers r_i and let μ be the smaller of the numbers ρ and 1. Then we can find positive constants B, E and H such that

$$\left|\sum_{j=1}^{n} b_{ij}(x)z_j(x) + A_{i1}(x)\right| \leqq B, \quad \left|\sum_{j=2}^{\infty} A_{ij}(x)p^{j-2}\right| \leqq E,$$

$$\left|\sum_{m=2}^{\infty}\Sigma a^{(i)}_{a_0,a_1,\cdots,a_n}(x)p^{m-2}z_1^{a_1}(x)\cdots z_n^{a_n}(x)\right| \leqq H,$$

for $|z_i(x)| \leqq r$, $|p| \leqq \mu$, $|x| \geqq K$.

We shall take λ to be the smaller of the numbers μ and $(1/3(E+H)) \cdot (r/4n)$. Also, take R to be the larger of K and $2B(4n/r)$. Then

$$|l_i(z_1(x), \cdots, z_n(x); p; x)| \leqq |p|(E+H) + (1/|x|)B$$
$$\leqq (1/3)(r/4n) + (1/2)(r/4n) < r/4n,$$

for $|z_i(x)| \leqq r$, $|p| \leqq \lambda$, $|x| \geqq R$.

It follows that the system of equations (7) has a solution of the form (4). Also, we know from the properties of the l_i and the method of computing the $C_i^{(k)}(x, p)$ that each term in the right members of equations (4) is analytic in p and continuous in x. Therefore, it follows that these solution-functions of equations (7) are analytic in p and continuous in x, since the right members of equations (4) converge absolutely and uniformly in x and p. Finally, we obtain from these results and equations (6) that the solution-functions thus obtained for equations (1) are analytic in p, continuous in x, and vanish identically for p equal to zero.

4. **Example** $(n=1)$. Consider the equation having the form

$$y(x+1) = x^{-1}b(x)y(x) + f(y(x); p; x), \quad f(0; 0; x) \equiv 0, \tag{8}$$

where

$$f = g(x, p) + h(y(x); p; x), \tag{9}$$

$$g(x, p) = \sum_{i=2}^{\infty} p^i, \tag{9a}$$

$$h(y(x); p; x) = \sum_{m=2}^{\infty} y^m(x). \tag{9b}$$

The power series (9a) and (9b) converge and the function $b(x)$ is bounded and continuous for all values of $y(x)$, p, and x for which

$$G_1: \quad |y(x)| \leqq r < 1; \quad |p| \leqq \rho < 1; \quad |x| \geqq K.$$

Let us make the transformation

$$y(x) = pz(x). \tag{10}$$

Then (8) is transformed into the equation

$$z(x+1) = x^{-1}b(x)z(x) + p\sum_{i=2}^{\infty} p^{i-2} + p\sum_{m=2}^{\infty} p^{m-2}z^m(x)$$
$$\equiv l(z(x); p; x).$$

The series $\sum_{m=2}^{\infty} z^m(x)$ converges for $|z(x)| \leqq r < 1$ and dominates the series $\sum_{m=2}^{\infty} p^{m-2} z^m(x)$, since $|p| < 1$. Also, the geometric series $\sum_{j=2}^{\infty} p^{j-2}$ converges for $|p| < 1$. Therefore, we can find positive constants B, E, and H such that

$$|b(x)z(x)| \leqq B, \quad \left|\sum_{j=2}^{\infty} p^{j-2}\right| \leqq E,$$

$$\left|\sum_{m=2}^{\infty} p^{m-2} z^m(x)\right| \leqq H,$$

for

$$|z(x)| \leqq r; \qquad |p| \leqq \rho; \qquad |x| \geqq K.$$

Take λ to be the smaller of the numbers ρ and $(1/3(E+H))(r/4)$. Also, take R to be the larger of the numbers K and $2B(4/r)$. Then

$$\begin{aligned} |l(z(x); p; x)| &\leqq (1/|x|)B + |p|(E + H) \\ &\leqq (r/2B(4))B + (1/3(E + H))(r/4)(E + H) < r/4, \end{aligned}$$

for

$$|z(x)| \leqq r; \qquad |p| \leqq \lambda; \qquad |x| \geqq R.$$

We have

$$C^{(1)}(x, p) = \sum_{j=2}^{\infty} p^{j-1} = p + p^2 + p^3 + \cdots,$$

$$C^{(2)}(x, p) = \frac{b(x-1)}{(x-1)}(p + p^2 + p^3 + \cdots)$$

$$C^{(3)}(x, p) = \frac{b(x-1)b(x-2)}{(x-1)(x-2)}(p + p^2 + \cdots) + p(p + p^2 + \cdots)^2,$$

. . . . . .

Therefore, we can write

$$z(x) = g_1(x)p + g_2(x)p^2 + \cdots,$$

where

$$\begin{aligned} g_1(x) = 1 &+ \frac{b(x-1)}{x-1} + \frac{b(x-1)b(x-2)}{(x-1)(x-2)} + \cdots \\ &+ \frac{b(x-1)\cdots b(x-n+1)}{(x-1)\cdots(x-n+1)} + \cdots. \end{aligned}$$

Finally, we have

$$y(x) = g_1(x)p^2 + g_2(x)p^3 + \cdots .$$

Direct substitution of this value for $y(x)$ in (8) shows that $g_1(x)$ must satisfy the nonhomogeneous linear difference equation

$$g_1(x+1) = 1 + \frac{b(x)}{x}\, g_1(x).$$

We have

$$\begin{aligned} g_1(x+1) &= 1 + \frac{b(x)}{x} + \frac{b(x)}{x}\,\frac{b(x-1)}{(x-1)} + \cdots \\ &\quad + \frac{b(x)}{x}\,\frac{b(x-1)\cdots b(x-n+2)}{(x-1)\cdots(x-n+2)} + \cdots \\ &= 1 + \frac{b(x)}{x}\left(1 + \frac{b(x-1)}{(x-1)} + \cdots \right. \\ &\quad \left. + \frac{b(x-1)\cdots b(x-n+1)}{(x-1)\cdots(x-n+1)} + \cdots\right) \\ &= 1 + \frac{b(x)}{x}\, g_1(x). \end{aligned}$$

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A THEOREM IN FINITE PROJECTIVE GEOMETRY AND AN APPLICATION TO STATISTICS[1]

ESTHER SEIDEN

R. C. Bose showed that certain statistical problems of design of experiments can be attacked fruitfully by interpreting the statistical terms involved in terms of finite geometries. In particular, the geometrical interpretation proved useful when applied to the problem of determining the maximum number of factors which can be accommodated in symmetrical factorial design without confounding the degrees of freedom belonging to interactions of a given order or lesser.

In a paper *Mathematical theory of factorial design* (Sankhya vol. 8 (1947) pp. 107–166) R. C. Bose proved that the maximum number, $m_t(r, s)$, of factors which can be accommodated in symmetrical factorial design in which each factor is at $s=p^n$ levels (p being a positive prime and n a positive integer) and each block is of size s^r, without confounding any degrees of freedom belonging to any interaction involving t or lesser number of factors, is given by the maximum number of points of a finite projective space $PG(r-1, s)$ (of r homogenous co-ordinates each of which is capable of s values) such that no t of the chosen points are conjoint (lie in a space of $t-2$ dimensions). Furthermore, R. C. Bose proved in the above mentioned paper that $m_3(4, s)=s^2+1$ when s is a power of an odd prime and $s^2+1 \leqq m_3(4, s) \leqq s^2+s+2$ when $s=2^n$, $n>1$. This inequality gives, for $s=4$, $17 \leqq m_3(4, 4) \leqq 22$. It is the purpose of this paper to prove that $m_3(4, 4)=17$. In other words: the maximum number of points of $PG(3, 4)$ such that no 3 of them are collinear cannot exceed 17 (Theorem 5).

In order to make the reading of this paper independent of the literature on finite geometry we explain shortly the $PG(3, 4)$. The points in $PG(3, 4)$ can be represented by attaching to it a Galois field with 4 elements, $GF(4)$, built with the help of an irreducible polynomial, $\epsilon^2+\epsilon+1=0$, where a point is understood to be an ordered quadruple of elements of $GF(4)$, not all zero. Two points (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) are defined to be identical if there exists a nonzero element of $GF(4)$, say ρ, such that $x_i=\rho y_i$, $i=1, 2, 3, 4$. Thus, the total number of points is seen to be $(4^4-1)/3=85$. By a line passing through two different points (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4)

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is understood the set of points $(\mu x_1+\lambda y_1, \mu x_2+\lambda y_2, \mu x_3+\lambda y_3, \mu x_4+\lambda y_4)$ where μ, λ are elements of $GF(4)$. It is seen that the number of points on a line is 5. Defining, similarly, what is meant by a plane, the number of planes passing through a line will be 5 and the number of points on a plane 21. Finally, the remark that the observations usual in projective geometry concerning the choice of the coordinate system hold true in $PG(3, 4)$.

The following two theorems are known to hold in $PG(3, 4)$.

THEOREM 1. *The three diagonal points of a nondegenerate plane quadrangle are distinct and lie on a line, say L.*

The proof follows by taking coordinates such that the points of the quadrangle are $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, $(0, 1, 1, 1)$. In this case the diagonal points are $(0, 1, 1, 0)$, $(0, 1, 0, 1)$, $(0, 0, 1, 1)$ and the equation of L becomes evidently $x_1=0$, $x_2+x_3+x_4=0$.

THEOREM 2. *In every plane there are 6 "no three collinear" points.*

$(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, $(0, 1, 1, 1)$, $(0, 1, \epsilon, \epsilon^2)$, $(0, 1, \epsilon^2, \epsilon)$ are six such points of the plane $x_1=0$. Evidently the last two points lie on L and differ from the above-mentioned three diagonal points. Moreover, together with them they exhaust the points of L.

THEOREM 3. *Any 4 no three collinear points Q_1, Q_2, Q_3, Q_4 in a plane determine uniquely 2 more points, say Q_5, Q_6, such that $Q_1, Q_2, \cdots, Q_6$ are no three collinear points.*

Let us consider the five lines in the plane passing through Q_1. They exhibit all the points of the plane. Three of them coincide with the lines determined by the pairs of points Q_1Q_2, Q_1Q_3, Q_1Q_4. Each of the remaining two lines meets the lines Q_2Q_3, Q_2Q_4, Q_3Q_4 in three distinct points. Thus on each of these two lines there remains just one point, say Q_5, Q_6, respectively which form together with $Q_1, \cdots, Q_4$ a set of no three collinear points.

DEFINITION. Any set of 6 no three collinear points of a plane will be called 6-basic.

THEOREM 4. *Consider two planes p_1 and p_2 intersecting in a line determined by two points U_1, U_2 and 4 more points Q_1, Q_2, Q_3, Q_4 of p_1 and R_1, R_2, R_3, R_4 of p_2 so that $U_1, U_2, Q_1, Q_2, Q_3, Q_4$ as well as $U_1, U_2, R_1, R_2, R_3, R_4$ are 6-basic. Then,*

1. *The 16 lines Q_iR_j $(i, j=1, 2, 3, 4)$ are distinct.*
2. *They can be distributed into four sets of 4 lines so that the 4 lines of each set intersect in exactly one point.*

3. *If Q_iQ_j and R_lR_m are coplanar, then they meet the line U_1U_2 in the same point.*

4. *The four points of intersection of the lines Q_iR_j, say O_1, O_2, O_3, O_4, lie on a plane, say p_3, passing through U_1U_2.*

5. *U_1, U_2, O_1, O_2, O_3, O_4 are 6-basic and the role of the sets of Q's or R's or O's can be interchanged.*

PROOF. 1. If a line λ given by Q_iR_j were identical with that given by Q_kR_l, then, assuming without loss of generality $i \neq k$, it would follow that points Q_i, Q_k, R_j lie on λ and consequently that R_j is in p_1 against the assumption.

2. There exists a two-to-two correspondence between the 6 lines Q_iQ_j and the 6 lines R_kR_l if we associate with line Q_iQ_j any of the two lines R_kR_l which pass through the same one of the three points of L other than U_1, U_2. By this correspondence the points of a given triangle $Q_iQ_jQ_s$ are made to correspond to those of an arbitrary triangle $R_kR_lR_m$ in a unique way. For Q_iQ_j corresponds to at most one side, say R_kR_l; Q_iQ_s corresponds to another side, say R_kR_m; hence the intersection Q_i of Q_iQ_j and Q_iQ_s corresponds to the intersection of R_kR_l and R_kR_m which is R_k; and accordingly Q_j must correspond to R_l, Q_s to R_m; it is seen that also line Q_jQ_s corresponds to line R_lR_m because these lines must pass through the remaining one of the three points of L other than U_1, U_2. Now we apply the converse of Desargue's theorem to the triangles $Q_iQ_jQ_s$ and $R_kR_lR_m$ and obtain that the lines Q_iR_k, Q_jR_l, Q_sR_m pass through one point, say O; furthermore, if Q' is the fourth point of the Q set, R' the fourth point of the R set, then $Q'R'$ passes also through O; for the triangle Q_iQ_jQ' corresponds to R_kR_lR' necessarily point by point in the same order since line Q_iQ' is different from Q_iQ_j and Q_iQ_s, hence these three lines intersect with L at three distinct points other than U_1, U_2, thus determining the intersection of L with Q_iQ' by exclusion, and the same holds for R_kR'. Applying the converse of Desargue's theorem to the triangles Q_iQ_jQ' and R_kR_lR', we find that Q_iR_k, Q_jQ_l, $Q'R'$ pass through a point which necessarily is O. Making any of the four triangles formed by 3 out of the four points R_1, R_2, R_3, R_4 correspond to the triangle Q_1, Q_2, Q_3 in the indicated manner, we obtain 4 points O_1, O_2, O_3, O_4 as points of perspectivity. They are distinct, for if, for example, the lines Q_1R_1, Q_2R_2, Q_3R_3, Q_4R_4 meet at O_1, then a line Q_iR_j, where $i \neq j$, cannot pass through O_1 because then R_i, R_j, Q_i would be collinear and consequently Q_i would lie on the plane p_2 against the assumption.

3. Two coplanar lines Q_iQ_j and R_lR_m meet the line L (intersection of the planes p_1 and p_2) in the same point, because otherwise the planes p_1 and p_2 would be identical against the assumption.

4. Given any two points of the set O_1, O_2, O_3, O_4, say O_1 and O_2, there exist two points Q_i, Q_j and two points R_l, R_m such that the line Q_iR_l meets Q_jR_m in O_1 and Q_iR_m meets Q_jR_l in O_2. Take Q_i arbitrary, determine R_l so that the line Q_iR_l passes through O_1; determine R_m so that Q_iR_m passes through O_2; then O_2R_l must pass through another Q, say Q_j. Q_i, Q_j, R_l, R_m form evidently a nondegenerate plane quadrangle and thus the line Q_iQ_j meets the line R_lR_m in a point P and, by Theorem 1, O_1, O_2, P are collinear. But P lies, by 3, on the line L, thus O_1, O_2, U_1, U_2 are coplanar. The same holds for any O_i, O_j, U_1, U_2, showing that all O_1, O_2, O_3, O_4 lie in a plane through U_1, U_2, say p_3.

5. Line U_1U_2 does not contain an O; furthermore any line O_iO_j meets the line U_1U_2 at a point P different from U_1 and U_2, so that no U lies on a line joining two O's and no O lies on a line joining two U's. It remains to show that no three of the O's are collinear. Assume without loss of generality that

O_1 is intersection of Q_1R_1 with Q_2R_2,

O_2 is intersection of Q_1R_2 with Q_2R_1,

O_3 is intersection of Q_1R_3 with Q_2R_4.

Now if O_1, O_2, O_3 were collinear, then R_1, R_2, R_3 would be obtained as intersections of their plane (which does not contain Q_1) with the plane $Q_1O_1O_2O_3$; hence R_1, R_2, R_3 would be collinear against assumption. Thus no three O's are collinear. It is seen that, under the assumption made above, Q_1 is the intersection of the lines O_1R_1, O_2R_2, O_3R_3, O_4R_4 and R_1 is the intersection of O_1Q_1, O_2Q_2, O_3Q_3, O_4Q_4. Evidently similar results follow for the remaining 3 groups of 4 lines O_iR_j or O_iQ_j respectively. Thus the rôles of the O's, Q's, and R's can be interchanged.

THEOREM 5. *The maximum number of points which can be chosen in* $PG(3, 4)$ *such that no three of the chosen points are collinear does not exceed* 17.

PROOF. Let C be the set consisting of the maximum number of points no three collinear in $PG(3, 4)$. Assume without loss of generality that U_1, U_2 belong to C. The proof is trivial in case when, among the 5 planes passing through U_1U_2, there do not exist 2 planes such that one of them contains 4 points, other than U_1, U_2, belonging to C and the other plane at least 3 such points, because in this case the total number of points belonging to C would readily be seen to be at most 17. Hence assume without loss of generality that Q_1, Q_2, Q_3, Q_4 as well as R_1, R_2, R_3 belong to C. Consider two cases (1) R_4 belongs

to C, (2) R_4 does not belong to C.

In case 1 the 16 distinct lines R_iQ_j $(i, j=1, 2, 3, 4)$ contain 16 points of each of the two planes, say p_4, p_5, other than p_1, p_2, p_3. Since the remaining 5 points of these planes belong to the line L, no points of these planes other than U_1, U_2 belong to C. C can contain at most 4 additional points of p_3; accordingly, the total number of points in C amounts to 14 $(2+3\times4=14)$.

In case 2, the points of p_4 and p_5 which belong to C are on the lines Q_iR_4 $(i=1, 2, 3, 4)$. Since the Q's belong to C by assumption, there are at most 4 more points of these lines belonging to C. In addition at most 4 points other than U_1, U_2 of p_3 belong to C. Thus in this case the total number of points belonging to C is at most 17 $(2+3+3\times4=17)$.

REMARK. It can be proved that in case 2 the total number of points belonging to C is 13.

Theorem 5 together with the inequality $m_3(4, 4)\geqq17$ proved by R. C. Bose gives the required equality, namely $m_3(4, 4)=17$.

It is a pleasure to record the author's indebtedness to Professor R. C. Bose for suggesting this problem and to Professor Hans Lewy for frequent stimulating discussions which helped to prepare this paper.

It was proved after the paper was submitted to the editors that every set of 17 no three collinear points in $PG(3, 4)$ forms a quadric surface.

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